Good Reductions of Shimura Varieties of Hodge Type in Arbitrary Unramified Mixed Characteristic, Part II

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ABSTRACT. We prove a conjecture of Milne pertaining to the existence of integral canonical models of Shimura varieties of abelian type in arbitrary unramified mixed characteristic $(0, p)$.

KEY WORDS: Shimura pairs and varieties, affine group schemes, abelian schemes, integral models, p-divisible groups, and F-crystals.

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Contents

1. Introduction

Let $p \in \mathbb{N}$ be an arbitrary prime. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at its prime ideal (p). In this paper we prove the existence of *integral canonical models* of *Shimura varieties* of *abelian type* in unramified mixed characteristic $(0, p)$ (i.e., over finite, étale $\mathbb{Z}_{(p)}$-algebras). In this introduction we first begin by recalling basic things on *Siegel modular varieties* and *Mumford moduli schemes*. Then we recall basic types of Shimura varieties and previous works on the existence of integral canonical models. We will end up the introduction by stating our main results and by outlining the strategy to prove them. Let $d \in \mathbb{N}$.

1.1. Siegel modular varieties and Mumford moduli schemes. Let (A, λ_A) be a principally polarized abelian variety over $\mathbb{C}$ of dimension d. Let $L := H_1(A(\mathbb{C}), \mathbb{Z})$ be the first homology group of the analytic space $A(\mathbb{C})$ with coefficients in $\mathbb{Z}$; it is a free abelian group of rank $2d$. Let $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$ be the perfect alternating form on L induced

naturally by λ_A. Let $W := L \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(A(\mathbb{C}), \mathbb{Q})$. The classical *Hodge theory* provides us with a Hodge decomposition

$$L \otimes_{\mathbb{Z}} \mathbb{C} = W \otimes_{\mathbb{Q}} \mathbb{C} = F^{-1,0} \oplus F^{0,-1} \tag{1}$$

such that under the standard complex conjugation of $W \otimes_{\mathbb{Q}} \mathbb{C}$ we have an identity $\overline{F^{-1,0}} = F^{0,-1}$. More precisely, one can identify $F^{-1,0} = \mathrm{Lie}(A)$ and $F^{0,-1} = \mathrm{Hom}(H^1(A, \mathcal{O}_A), \mathbb{C})$, where $\mathcal{O}_A$ is the structured ring sheaf on A. Both $F^{-1,0}$ and $F^{0,-1}$ are isotropic with respect to ψ and in fact $2\pi i\psi$ is a polarization of the Hodge $\mathbb{Z}$-structure on L defined by (1). Thus to (1) corresponds naturally a homomorphism

$$x_A : \mathbb{C}^* \to \mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$$

of reductive groups over $\mathbb{R}$. Let $\mathcal{S}$ be the $\mathbf{GSp}(W, \psi)(\mathbb{R})$-conjugacy class of x_A. If C_A is the centralizer of x_A in $\mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$, then we have $\mathcal{S} = \mathbf{GSp}(W \otimes_{\mathbb{R}} \mathbb{R}, \psi)(\mathbb{R})/C_A(\mathbb{R})$ and thus $\mathcal{S}$ gets a natural structure of a hermitian symmetric domain isomorphic to two copies of the Siegel space of genus d. The pair $(\mathbf{GSp}(W, \psi), \mathcal{S})$ is called a *Siegel modular pair* and it is the most studied type of *Shimura pairs*. We have a canonical identification

$$A(\mathbb{C}) = F^{0,-1} \backslash (W \otimes_{\mathbb{Q}} \mathbb{R})/L \tag{2}$$

of analytic complex Lie groups. If (B, λ_B) is another principally polarized abelian variety over $\mathbb{C}$ of dimension d, then there exists an element $g \in \mathbf{GSp}(W, \psi)(\mathbb{R})$ such that the analytic complex Lie group $B(\mathbb{C})$ is isomorphic to $g(F^{0,-1}) \backslash (W \otimes_{\mathbb{Q}} \mathbb{R})/L$ in such a way that the perfect alternating form on L defined by λ_B is either ψ or $-\psi$. One easily gets that the course moduli space of principally polarized abelian varieties over $\mathbb{C}$ of dimension d is isomorphic to

$$\mathbf{GSp}(L, \psi)(\mathbb{Z}) \backslash \mathcal{S} \tag{3a}$$

(see [BB, Thm. 10.11] for the canonical structure of (3a) as a normal, quasi-projective variety over $\mathbb{C}$). If $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles of $\mathbb{Q}$, then (3a) is isomorphic to

$$\mathbf{GSp}(W, \psi)(\mathbb{Q}) \backslash [\mathcal{S} \times (\mathbf{GSp}(W, \psi)(\mathbb{A}_f)/\mathbf{GSp}(L, \psi)(\widehat{\mathbb{Z}}))]. \tag{3b}$$

Let $N \geq 3$ be an integer relatively prime to p. Let

$$K(N) := \{g \in \mathbf{GSp}(L, \psi)(\widehat{\mathbb{Z}}) | g \text{ mod } N \text{ is identity}\} \text{ and } K_p := \mathbf{GSp}(L, \psi)(\mathbb{Z}_p).$$

Let $\mathcal{A}_{d,1,N}$ be the *Mumford moduli scheme* over $\mathbb{Z}[\frac{1}{N}]$ that parameterizes isomorphism classes of principally polarized abelian schemes over $\mathbb{Z}[\frac{1}{N}]$-schemes that are of relative dimension d and that are endowed with a symplectic similitude level-N structure (cf. [MFK, Thms. 7.9 and 7.10] applied to symplectic similitude level structures instead of

only to level structures). The $\mathbb{Z}[\frac{1}{N}]$-scheme $\mathcal{A}_{d,1,N}$ is smooth and quasi-projective, cf. loc. cit. Similarly to (3b) one gets a natural identification

$$\mathcal{A}_{d,1,N}(\mathbb{C}) = \mathbf{GSp}(W,\psi)(\mathbb{Q})\backslash[\mathcal{S} \times (\mathbf{GSp}(W,\psi)(\mathbb{A}_f)/K(N))]. \tag{4}$$

Based on this and on the classical works of Shimura, Taniyama, etc., one gets an identification

$$\mathcal{A}_{d,1,N,\mathbb{Q}} = \mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})/K(N) \tag{5}$$

of $\mathbb{Q}$–schemes, where $\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})$ is the *canonical model* as defined in [De1] of the complex Shimura variety

$$\mathrm{Sh}(\mathbf{GSp}(W,\psi),\mathcal{S})_{\mathbb{C}} = \mathrm{proj.lim.}_{K\in\Sigma(G)}\mathbf{GSp}(W,\psi)(\mathbb{Q})\backslash[\mathcal{S} \times (\mathbf{GSp}(W,\psi)(\mathbb{A}_f)/K)].$$

Here $\Sigma(G)$ is the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation. Thus

$$\mathcal{M} := \mathrm{proj.lim.}_{N\,\geq\,3,(N,p)=1}\mathcal{A}_{d,1,N}$$

is a $\mathbb{Z}_{(p)}$-scheme which is an integral model of $\mathbf{GSp}(W,\psi)(\mathbb{Q})\backslash[\mathcal{S} \times (\mathbf{GSp}(W,\psi)(\mathbb{A}_f)/K_p)]$ over $\mathbb{Z}_{(p)}$. In fact $\mathcal{M}$ is the integral canonical model of $\mathbf{GSp}(W,\psi)(\mathbb{Q})\backslash[\mathcal{S}\times(\mathbf{GSp}(W,\psi)(\mathbb{A}_f)/K_p)]$ over $\mathbb{Z}_{(p)}$ in the sense of [Va1, Def. 3.2.3 6)] (see [Va1, Example 3.2.9] or [Mi2, Thm. 2.10]) and thus also in the sense of [Moo]. From this and [Va3, Thm. 1.3] one gets that the regular, formally smooth $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}$ is uniquely determined by its generic fibre $\mathcal{M}_{\mathbb{Q}}$ and by the following universal property:

(*) *if Z is a regular, formally smooth scheme over $\mathbb{Z}_{(p)}$, then each morphism $Z_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}}$ extends uniquely to a morphism $Z \to \mathcal{M}$ of $\mathbb{Z}_{(p)}$-schemes.*

The goal of this paper is to generalize the existence and the uniqueness of $\mathcal{M}$ to the case of all Shimura varieties of abelian type (i.e., to all Shimura varieties that are moduli spaces of polarized abelian motives endowed with level structures and motivic tensors).

1.2. Types of Shimura pairs. Let G be a reductive subgroup of $\mathbf{GSp}(W,\psi)$ for which any one of the following two equivalent statements holds:

(i) no simple compact factor of the adjoint group G^{ad} of G becomes compact over $\mathbb{R}$ and there exists an element $x \in \mathcal{S}$ which factors through $G_{\mathbb{R}}$;

(ii) there exists an element $x \in \mathcal{S}$ which factors through $G_{\mathbb{R}}$ and $G^{\mathrm{ad}}_{\mathbb{R}}$ is generated by the $G(\mathbb{R})$-conjugates of the homomorphism $x^{\mathrm{ad}} : \mathbb{C}^* \to G^{\mathrm{ad}}_{\mathbb{R}}$ defined naturally by $x : \mathbb{C}^* \to G_{\mathbb{R}}$.

Let $\mathcal{X}$ be the $G(\mathbb{R})$-conjugacy class of an element $x \in \mathcal{S}$ that factors through $G_{\mathbb{R}}$. The pair $(G,\mathcal{X})$ is a Shimura pair in the sense of [De2] and we have an injective map $f : (G,\mathcal{X}) \to (\mathbf{GSp}(W,\psi),\mathcal{S})$ of Shimura pairs. A Shimura pair is called of *Hodge type* if it is isomorphic to a Shimura pair of the form $(G,\mathcal{X})$ (for some $d \in \mathbb{N}$). Let $\mathcal{X}^{\mathrm{ad}}$ be the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of the homomorphism $x^{\mathrm{ad}} : \mathbb{C}^* \to G^{\mathrm{ad}}_{\mathbb{R}}$. The pair $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ is a Shimura pair called the *adjoint Shimura pair* of $(G,\mathcal{X})$.

A Shimura pair $(G, \mathcal{X})$ of Hodge type is called of *PEL type*, if there exists an injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that G is the identity component of the intersection of $\mathbf{GSp}(W, \psi)$ with the double centralizer of G in $\mathbf{GL}_W$. Here PEL stands for polarizations, endomorphisms, and level structures. In such a case, we say that $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ is a *PEL type embedding*. If moreover, G itself is the intersection of $\mathbf{GSp}(W, \psi)$ with the double centralizer of G in $\mathbf{GL}_W$, then we say that the Shimura pair $(G, \mathcal{X})$ is of *moduli PEL type* and we say that $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ is a *moduli PEL type embedding*.

A Shimura pair $(G_1, \mathcal{X}_1)$ is called of *preabelian type* if there exists a Shimura pair $(G, \mathcal{X})$ of Hodge type such that we have an isomorphism $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) \xrightarrow{\sim} (G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})$ of adjoint Shimura pairs. If moreover, the isomorphism $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) \xrightarrow{\sim} (G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})$ is defined by an isogeny $G^{\mathrm{der}} \to G_1^{\mathrm{der}}$ between the derived groups of G and G_1, then we say that $(G_1, \mathcal{X}_1)$ is of *abelian type*.

We say $(G_1, \mathcal{X}_1)$ is *unitary*, if all simple factors of $G_{\mathbb{C}}^{\mathrm{ad}}$ are $\mathbf{PGL}$ groups. Following [Va5, Def. 1.1] we say $(G_1, \mathcal{X}_1)$ *has compact factors*, if for each simple factor G_0 of G_1^{ad} there exists a simple factor of $G_{0,\mathbb{R}}$ which is compact.

Let $\mathrm{Sh}(G_1, \mathcal{X}_1)$ be the canonical model of $(G_1, \mathcal{X}_1)$ over the reflex field $E(G_1, \mathcal{X}_1)$ of $(G_1, \mathcal{X}_1)$ (see [Di1], [De2], [Mi1], and [Mi4]). The natural closed embedding of complex spaces $\mathcal{X} \times G(\mathbb{A}_f) \hookrightarrow \mathcal{S} \times \mathbf{GSp}(W, \psi)(\mathbb{A}_f)$ gives birth naturally via passage to quotients to a closed embedding $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}} \hookrightarrow \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{\mathbb{C}}$. The reflex field $E(G, \mathcal{X})$ is a number field which is the smallest subfield of $\mathbb{C}$ with the property that the last closed embedding is the pull back of a closed embedding

$$\mathrm{Sh}(G, \mathcal{X}) \hookrightarrow \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{E(G,\mathcal{X})} \tag{6}$$

(see [De1, Cor. 5.4]). Let $J_p := K_p \cap G(\mathbb{Q}_p)$. As we have $\mathrm{Sh}(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f))$ and $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})(\mathbb{C}) = \mathbf{GSp}(W, \psi)(\mathbb{Q}) \backslash (\mathcal{S} \times \mathbf{GSp}(W, \psi)(\mathbb{A}_f))$ (see [De2, Cor. 2.1.11]), it is easy to see that (6) induces naturally a closed embedding homomorphism

$$\mathrm{Sh}(G, \mathcal{X})/J_p \hookrightarrow \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{E(G,\mathcal{X})}/K_p. \tag{7}$$

1.3. Integral canonical models. Let $(G_1, \mathcal{X}_1)$ be a Shimura pair of abelian type such that the group $G_{1,\mathbb{Q}_p}$ is unramified (i.e., it has a Borel subgroup and it splits over a finite, unramified extension of $\mathbb{Q}_p$). We recall that this equivalent to the fact $G_{1,\mathbb{Q}_p}$ extends to a reductive group scheme $G_{1,\mathbb{Z}_p}$ over $\mathbb{Z}_p$, cf. [Ti2]. Each compact, open subgroup of $G_1(\mathbb{Q}_p) = G_{1,\mathbb{Q}_p}(\mathbb{Q}_p)$ of the form $H_1 := G_{1,\mathbb{Z}_p}(\mathbb{Z}_p)$ is called *hyperspecial*. We refer to the triple $(G_1, \mathcal{X}_1, H_1)$ as a *Shimura triple of abelian type* (with respect to p). By a map $q : (G_1, \mathcal{X}_1, H_1) \to (\tilde{G}_1, \tilde{\mathcal{X}}_1, \tilde{H}_1)$ of Shimura triples of abelian type (with respect to p) we mean a map $q : (G_1, \mathcal{X}_1) \to (\tilde{G}_1, \tilde{\mathcal{X}}_1)$ of Shimura pairs such that the homomorphism $q(\mathbb{Q}_p) : G_1(\mathbb{Q}_p) \to \tilde{G}_1(\mathbb{Q}_p)$ maps H_1 to $\tilde{H}_1$.

As the group $G_{1,\mathbb{Q}_p}$ is unramified, the reflex field $E(G_1, \mathcal{X}_1)$ is unramified above p (cf. [Mi3, Cor. 4.7 (a)]). Thus the normalization $E(G_1, \mathcal{X}_1)_{(p)}$ of $\mathbb{Z}_{(p)}$ in $E(G_1, \mathcal{X}_1)$ is a finite, étale $\mathbb{Z}_{(p)}$-algebra. Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of $\mathbb{Q}$ with the p-component omitted; we have $\mathbb{A}_f = \mathbb{A}_f^{(p)} \times \mathbb{Q}_p$.

1.3.1. Definitions. **(a)** By an *integral model* of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ over $E(G_1, \mathcal{X}_1)_{(p)}$ we mean a faithfully flat scheme $\mathcal{N}_1$ over $E(G_1, \mathcal{X}_1)_{(p)}$ together with a continuous right action of $G_1(\mathbb{A}_f^{(p)})$ on it in the sense of [De2, Subsubsect. 2.7.1], such that there exists a $G_1(\mathbb{A}_f^{(p)})$-equivariant isomorphism

$$\mathcal{N}_{1,E(G_1,\mathcal{X}_1)} \xrightarrow{\sim} \mathrm{Sh}(G_1, \mathcal{X}_1).$$

The integral model $\mathcal{N}_1$ is said to be *smooth* (resp. *normal*) if there exists a compact, open subgroup $\tilde{H}$ of $G_1(\mathbb{A}_f^{(p)})$ such that for every inclusion $\tilde{H}_2 \subseteq \tilde{H}_1$ of compact, open subgroups of $\tilde{H}$, the natural morphism $\mathcal{N}_1/\tilde{H}_2 \to \mathcal{N}_1/\tilde{H}_1$ is a finite, étale morphism between smooth schemes (resp. between normal schemes) of finite type over $E(G_1, \mathcal{X}_1)_{(p)}$.

We say that $\mathcal{N}_1$ is *quasi-projective* or *projective* if we can choose $\tilde{H}$ such that $\mathcal{N}_1/\tilde{H}$ is a quasi-projective or projective (respectively), smooth $E(G_1, \mathcal{X}_1)_{(p)}$-scheme.

(b) A regular, faithfully flat $O_{(v)}$-scheme Y is called *healthy* regular, if for each open subscheme U of Y which contains $Y_{\mathbb{Q}}$ and all points of Y of codimension 1, every abelian scheme over U extends to an abelian scheme over Y.

(c) A scheme Z over $E(G_1, \mathcal{X}_1)_{(p)}$ is said to have the *extension property* if for each healthy regular scheme Y over $E(G_1, \mathcal{X}_1)_{(p)}$, every $E(G_1, \mathcal{X}_1)$-morphism $Y_{E(G_1,\mathcal{X}_1)} \to Z_{E(G_1,\mathcal{X}_1)}$ extends uniquely to an $E(G_1, \mathcal{X}_1)_{(p)}$-morphism $Y \to Z$.

(d) A smooth integral model of $\mathrm{Sh}(G_1, \mathcal{X}_1)$ over $E(G_1, \mathcal{X}_1)_{(p)}$ that has the extension property is called an *integral canonical model* of $(G_1, \mathcal{X}_1, H_1)$ (or of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ over $E(G_1, \mathcal{X}_1)_{(p)}$).

1.3.2. Two important properties. Every regular, formally smooth scheme over $E(G, \mathcal{X})_{(p)}$ is a healthy regular scheme, cf. [Va3, Thm. 1.3]. From this and Yoneda Lemma we get that:

(a) an integral canonical model of $(G_1, \mathcal{X}_1, H_1)$ is uniquely determined up to a canonical isomorphism;

(b) if we have a map $q : (G_1, \mathcal{X}_1, H_1) \to (\tilde{G}_1, \tilde{\mathcal{X}}_1, \tilde{H}_1)$ of Shimura triples of abelian type and if the integral canonical models $\mathcal{N}_1$ and $\tilde{\mathcal{N}}_1$ of $(G_1, \mathcal{X}_1, H_1)$ and $(\tilde{G}_1, \tilde{\mathcal{X}}_1, \tilde{H}_1)$ (respectively) exist, then the natural morphism $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1 \to \mathrm{Sh}(\tilde{G}_1, \tilde{\mathcal{X}}_1)/\tilde{H}_1$ of $E(\tilde{G}_1, \tilde{\mathcal{X}}_1)$-schemes extends uniquely to a *functorial morphism* $\mathcal{N}_1 \to \tilde{\mathcal{N}}_1$ of $E(\tilde{G}_1, \tilde{\mathcal{X}}_1)_{(p)}$-schemes.

The next Proposition is a $\mathbb{Z}_{(p)}$-variant of the classical results of Satake and Deligne (see [Sa1], [Sa2], and [De2, Prop. 2.3.10]) which is only a slightly more general form of [Va1, Thm. 6.5.1.1 a) to c)].

1.4. Proposition. *Let $(G_1, \mathcal{X}_1)$ be a simple, adjoint Shimura pair of abelian type. We assume that the group $G_{1,\mathbb{Q}_p}$ is unramified. Let H_1 be a hyperspecial subgroup of $G_1(\mathbb{Q}_p)$ i.e., be the group of $\mathbb{Z}_p$-valued points of a reductive group scheme $G_{1,\mathbb{Z}_p}$ over $\mathbb{Z}_p$ that extends $G_{1,\mathbb{Q}_p}$. Then there exists an injective map $f : (G, \mathcal{X}) \to (\boldsymbol{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that the following four properties hold:*

(i) *we have a natural identification $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}) = (G_1, \mathcal{X}_1)$;*

(ii) *there exists a $\mathbb{Z}$-lattice L of W which is self dual with respect to ψ and for which the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\boldsymbol{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$ is a reductive group scheme over $\mathbb{Z}_{(p)}$;*

(iii) *the natural quotient homomorphism* $G_{\mathbb{Q}_p} \twoheadrightarrow G^{\mathrm{ad}}_{\mathbb{Q}_p} = G_{1,\mathbb{Q}_p}$ *extends uniquely to a quotient homomorphism* $G_{\mathbb{Z}_p} \twoheadrightarrow G^{\mathrm{ad}}_{\mathbb{Z}_p} = G_{1,\mathbb{Z}_p}$, *where* $G_{\mathbb{Z}_p} := G_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$;

(iv) *the semisimple group cover* G^{der} *of* G_1 *is the maximal one allowed by the abelian type (i.e., if* $(G_2, \mathcal{X}_2)$ *is any other Shimura pair of abelian type whose adjoint Shimura pair is* $(G_1, \mathcal{X}_1)$, *then the isogeny* $G^{\mathrm{der}} \to G_1$ *factors through the isogeny* $G_2^{\mathrm{der}} \to G_1$*).*

1.5. Theorem. *Under the hypotheses of Proposition 1.4, we can choose the injective map of Shimura pairs* $f : (G, \mathcal{X}) \to (\boldsymbol{GSp}(W, \psi), \mathcal{S})$ *such that the properties 1.4 (i) to (iii) hold and moreover the following three additional properties hold as well:*

(i) *if* $\mathcal{N}$ *is the normalization of the Zariski closure of* $\mathrm{Sh}(G, \mathcal{X})/J_p$ *in* $\mathcal{M}_{E(G,\mathcal{X})_{(p)}}$ *(this makes sense due to (7)), then* $\mathcal{N}$ *is the integral canonical model of* $(G, \mathcal{X}, J_p)$ *and is quasi-projective (here* $J_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ *is as in the end of Subsection 1.2);*

(ii) *the integral canonical model* $\mathcal{N}_1$ *of* $(G_1, \mathcal{X}_1, H_1)$ *exists and is quasi-projective;*

(iii) *the functorial morphism* $\mathcal{N} \to \mathcal{N}_1$ *of* $E(G_1, \mathcal{X}_1)_{(p)}$*-schemes, is a pro-étale cover of an open closed subscheme of* $\mathcal{N}_1$.

1.6. Theorem. *Let* $(G_1, \mathcal{X}_1, H_1)$ *be a Shimura triple of abelian type with respect to* p. *Then the following four properties hold:*

(a) *The integral canonical model* $\mathcal{N}_1$ *of* $(G_1, \mathcal{X}_1, H_1)$ *exists and it is quasi-projective.*

(b) *Let* $(G_1, \mathcal{X}_1, H_1) \to (G_2, \mathcal{X}_2, H_2)$ *be a map of Shimura triples with respect to* p *such that at the level of derived groups it induces an isogeny* $G_1^{\mathrm{der}} \to G_2^{\mathrm{der}}$. *The functorial morphism* $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1 \to \mathrm{Sh}(G_2, \mathcal{X}_2)/H_2$ *of* $E(G_2, \mathcal{X}_2)$*-schemes extends uniquely to a morphism* $q_1 : \mathcal{N}_1 \to \mathcal{N}_2$ *of* $E(G_2, \mathcal{X}_2)_{(p)}$*-schemes, where* $\mathcal{N}_2$ *is the integral canonical model of* $(G_2, \mathcal{X}_2, H_2)$. *Then* q_1 *is a pro-étale cover of an open closed subscheme of* $\mathcal{N}_{2,E(G_2,\mathcal{X}_2)_{(p)}}$.

(c) *Let* $(G_1, \mathcal{X}_1, H_1) \hookrightarrow (G_2, \mathcal{X}_2, H_2)$ *be an injective map of Shimura triples of abelian type. Let* $\mathcal{N}_1 \to \mathcal{N}_2$ *be the functorial morphism of* $E(G_2, \mathcal{X}_2)_{(p)}$*-schemes, where* $\mathcal{N}_2$ *is the integral canonical model of* $(G_1, \mathcal{X}_1, H_1)$. *Then* $\mathcal{N}_1$ *is the normalization of the Zariski closure of* $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ *in* $\mathcal{N}_2$.

(d) *Let* $(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}}) = \prod_{i \in I} (G_i, \mathcal{X}_i)$ *be the product decomposition into simple, adjoint Shimura pairs. We also assume that for each element* $i \in I$ *one of the following three conditions holds:*

(d.i) *there exists a prime* $q \in \mathbb{N}$ *such that the group* $G_{i,\mathbb{Q}_q}$ *is isotropic (i.e.,* $\mathbb{G}_m$ *is not a subgroup of it);*

(d.ii) *the Shimura pair* $(G_i, \mathcal{X}_i)$ *has compact factors in the sense of Subsection 1.2;*

(d.iii) *the Shimura pair* $(G_i, \mathcal{X}_i)$ *is unitary of strong compact type in the sense of [Va6, Def. 2.2.2 (b)].*

Then $\mathcal{N}_1$ *is projective.*

1.7. Previous results. For the sake of clarity, the previous results pertaining to Theorems 1.5 and 1.6 will be listed chronologically.

(i) Mumford proved the existence of integral canonical models of Siegel modular varieties. More precisely, the $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}$ together with the natural action of $\mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$ on it, is an integral canonical model of $(\mathbf{GSp}(W,\psi),\mathcal{S},K_p)$. Artin's method can be used to regain this result (see [Ar1], [Ar2], and [FC, Ch. I, Subsect. 4.11]).

(ii) If $f:(G,\mathcal{X})\to(\mathbf{GSp}(W,\psi),\mathcal{S})$ is a moduli PEL type embedding and if the condition 1.4 (ii) holds, then Zink proved that the Zariski closure of $\mathrm{Sh}(G,\mathcal{X})/J_p$ in $\mathcal{M}_{E(G,\mathcal{X})_{(p)}}$ (this makes sense due to (7)) is the integral canonical model of $(G,\mathcal{X},J_p)$ (see [Zi]). This result was reobtained in [LR].

(iii) If $f:(G,\mathcal{X})\to(\mathbf{GSp}(W,\psi),\mathcal{S})$ is a PEL type embedding, if the condition 1.4 (ii) holds, and if $p>2$, then Kottwitz pointed out that the arguments of [LR] can be used to get as well that the Zariski closure of $\mathrm{Sh}(G,\mathcal{X})/J_p$ in $\mathcal{M}_{E(G,\mathcal{X})_{(p)}}$ is the integral canonical model of $(G,\mathcal{X},J_p)$ (see [Ko]).

(iv) In [Va1] it is proved that Theorem 1.5 and Theorem 1.6 (a) to (c) hold provided $p>3$.

(v) See [Va3, Thm. 1.3] and [Va6, Thm. 1.3] for two corrections to [Va1] in connection to (iv). More precisely:

- the original argument of Faltings in [Va1, Subsubsect. 3.2.17, Step B, last paragraph] was incorrect and this has been corrected by [Va3, Prop. 4.1];
- the proof of Theorem 1.6 (a) for $p>3$ and for the cases when $G^{\mathrm{ad}}_{1,\mathbb{C}}$ has simple factors isomorphic to $\mathbf{PGL}_{pm}$ for some $m\in\mathbb{N}$ was partially incorrect in [Va1]; this has been corrected by [Va6, Thm. 1.3] (cf. [Va6, Appendix, E.3]).

(vi) In [Va6] it is proved that Theorems 1.5 and 1.6 (a) hold provided $(G_1,\mathcal{X}_1)$ is a unitary Shimura pair.

(vii) Theorem 1.6 (d) is only a direct consequence of previous results of Morita and Paugam (see [Mo] and [Pa]) and us (see [Va5] and [Va6]).

(viii) In [Va8] it is shown that Kottwitz's result (see (iii)) holds even if $p=2$.

(ix) In [Va9] it is proved that Theorems 1.5 and 1.6 hold if $p>2$ and if the adjoint Shimura pair $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ has compact factors.

(x) A 10 pages note of Kisin claims Theorem 1.6 (a) for $p>2$ (see Kisin's home page at U of Chicago). This note does not bring any new conceptual ideas to [Va1], [Va6], [Va7], and [Va9]. In fact, the note is only a variation of [Va1], [Va6], [Va7], and [Va9]. This variation is made possible due to recent advances in the theory of crystalline representations achieved by Fontaine, Breuil, Berger, and Kisin. The note also claims (without any details) that in the definition of $\mathcal{N}$ in the property 1.5 (i) one does not need to take the normalization.

1.8. On the strategy to prove Theorems 1.5 and 1.6. Theorem 1.6 is a direct consequence of Theorem 1.5 and of the methods developed in [Va1] to [Va6]. Thus we will detail here only on the strategy to prove Theorem 1.5.

It is known that $(G_1,\mathcal{X}_1)$ is of one of the following five disjoint types: A_n (with $n\geq 1$), B_n (with $n\geq 3$), C_n (with $n\geq 2$), $D_n^{\mathbb{H}}$ (with $n\geq 4$), and $D_n^{\mathbb{R}}$ (with $n\geq 4$). These

types were introduced in [De2]. For instance, $(G_1, \mathcal{X}_1)$ is of A_n, B_n, or C_n type if and only if all simple factors of $G_{1,\mathbb{C}}$ are of A_n, B_n, or C_n (respectively) Lie type. As explained in [Va9, Subsect. 1.9], to prove Theorem 1.5 one considers three disjoint cases:

(PELNONCOMP) all simple factors of $G_{1,\mathbb{R}}$ are non-compact and $(G_1, \mathcal{X}_1)$ is of either A_n (with $n \geq 1$) or C_n (with $n \geq 2$) or $D_n^{\mathbb{H}}$ (with $n \geq 4$) type;

(COMP) there exists a simple factor of $G_{1,\mathbb{R}}$ which is compact (i.e., the Shimura pair $(G_1, \mathcal{X}_1)$ has compact factors);

(SPINNONCOMP) all simple factors of $G_{1,\mathbb{R}}$ are non-compact and $(G_1, \mathcal{X}_1)$ is of either B_n (with $n \geq 3$) or $D_n^{\mathbb{R}}$ (with $n \geq 4$) type.

The proof of Theorem 1.5 in the (PELNONCOMP) case relies on the results 1.7 (ii), (iii), and (viii) and its essence is very well documented in the literature (see [Va6] for the A_n type; the C_n and $D_n^{\mathbb{H}}$ type cases are similar).

The proof of Theorems 1.5 (i) in the (COMP) case is presented in [Va9] for $p \geq 3$. Loc. cit. also got a weaker version of this result for $p = 2$ and mentioned the strategy to complete the proof for $p = 2$ as well. This strategy is brought to fruition here, cf. Appendix and Subsection 3.7. More precisely, in the (COMP) case with $p = 2$ we show that a motivic conjecture of Milne holds for the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$.

The proof of Theorem 1.5 (i) in the (SPINNONCOMP) case involves four ideas (see Subsection 3.8). We list them here using bullets.

• It is well known that $E(G_1, \mathcal{X}_1) = \mathbb{Q}$ and that moreover one can choose the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ such that we have as well $E(G, \mathcal{X}) = \mathbb{Q}$ (see [De2]).

• As $E(G, \mathcal{X}) = \mathbb{Q}$, a standard versal argument involving F-isocrystals shows that we can choose the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ such that the ordinary locus of the special fibre $\mathcal{L}$ of $\mathcal{N}$ is Zariski dense in $\mathcal{L}$.

• It is well known that the ordinary points of $\mathcal{N}$ belong to the regular, formally smooth locus of $\mathcal{N}$ over $\mathbb{Z}_{(p)}$ (see [No]). Moreover, it is easy to see that the mentioned motivic conjecture of Milne holds for ordinary points.

• Based on [Va9] (and its refinement for $p = 2$ presented in the Appendix), one gets that the intersection of the smooth locus of $\mathcal{N}$ with $\mathcal{L}$ is an open closed subscheme of $\mathcal{L}$ and therefore (due to the previous two bullets) it is $\mathcal{L}$ itself. From this one easily gets that Theorem 1.5 (i) holds in the (SPINNONCOMP) case.

The passage from Theorem 1.5 (i) to Theorem 1.5 (ii) and (iii) in the (COMP) and (SPINNONCOMP) cases is the same as in [Va1] if $p > 2$ and it is very much the same as in [Va6] if $p = 2$. More precisely, (regardless of what the prime p is) we show that we can choose the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ such that it factors through an injective map $f_2 : (G_2, \mathcal{X}_2) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ with the properties that: (a) $(G_2^{\mathrm{ad}}, \mathcal{X}_2^{\mathrm{ad}})$ is an adjoint, unitary Shimura pair, and (b) the resulting injective map $(G, \mathcal{X}) \hookrightarrow (G_2, \mathcal{X}_2)$ induces an injective map $(G_1, \mathcal{X}_1) \hookrightarrow (G_2^{\mathrm{ad}}, \mathcal{X}_2^{\mathrm{ad}})$ at the level of adjoint Shimura pairs. Using this and [Va6] one gets that Theorem 1.5 (ii) and (iii) follows from Theorem 1.5 (i).

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2. Preliminaries

Subsection 2.1 recalls standard notations on reductive group schemes. In Subsection 2.2 we prove Proposition 1.4. Subsection 2.3 recalls the notion of a cover between Shimura triples of abelian type. Proposition 2.4 is the very essence of Theorem 1.6 (c).

2.1. Reductive group schemes. A group scheme $\mathcal{R}$ over an affine scheme $\mathrm{Spec}(R)$ is called *reductive*, if it is smooth and affine and its fibres are connected and have trivial unipotent radicals (cf. [DG, Vol. III, Exp. XIX, 2.7]). Let $\mathcal{R}^{\mathrm{der}}$, $Z(\mathcal{R})$, $\mathcal{R}^{\mathrm{ab}}$, and $\mathcal{R}^{\mathrm{ad}}$ be the *derived group scheme* of $\mathcal{R}$, the *center* of $\mathcal{R}$, the *maximal commutative quotient* of $\mathcal{R}$, and the *adjoint group scheme* of $\mathcal{R}$ (respectively). Thus we have $\mathcal{R}^{\mathrm{ad}} = \mathcal{R}/Z(\mathcal{R})$ and $\mathcal{R}^{\mathrm{ab}} = \mathcal{R}/\mathcal{R}^{\mathrm{der}}$, cf. [DG, Vol. III, Exp. XXII, 4.3.6 and 6.2.1]. For a finite, étale morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_0)$, let $\mathrm{Res}_{R/R_0}\mathcal{H}$ be the reductive group scheme over $\mathrm{Spec}(R_0)$ obtained from F through the *Weil restriction of scalars* (see [BLR, Ch. 7, Sect. 7.6] and [Va2, Subsect. 2.3]).

If M is a free module of finite rank over a commutative $\mathbb{Z}$-algebra, then let $M^* := \mathrm{Hom}(M, R)$ and let $\mathcal{T}(M) := \oplus_{s,t\in\mathbb{N}\cup\{0\}} M^{\otimes s} \otimes_R M^{*\otimes t}$.

2.2. Proof of 1.4. Let $(G_1, \mathcal{X}_1)$ be a simple, adjoint Shimura pair of abelian type. We assume that the group $G_{1,\mathbb{Q}_p}$ is unramified. Let H_1 be a hyperspecial subgroup of $G_1(\mathbb{Q}_p)$ i.e., the group of $\mathbb{Z}_p$-valued points of a reductive group scheme $G_{1,\mathbb{Z}_p}$ over $\mathbb{Z}_p$ that extends $G_{1,\mathbb{Q}_p}$. In this Subsection we prove Proposition 1.4. Let F_1 be a number field such that we have an isomorphism $G_1 \tilde{\to} \mathrm{Res}_{F_1/\mathbb{Q}} G_1^{F_1}$, where $G_1^{F_1}$ is an absolutely simple adjoint group over F_1 (see [Ti1, Subsubsect. 3.1.2]). The number field F_1 is uniquely determined up to $\mathrm{Gal}(\mathbb{Q})$-conjugation (i.e., up to isomorphism).

As $G_{1,\mathbb{R}}$ is an inner form of its compact form (cf. [De2, p. 255]), it is a product of absolutely simple, adjoint groups over $\mathbb{R}$. This implies that the number field F_1 is totally real. As $G_{1,\mathbb{Q}_p}$ splits over an unramified extension F_p of $\mathbb{Q}_p$, the F_p-algebra $F_1 \otimes_{\mathbb{Q}} F_p$ is isomorphic to $F_p^{[F:\mathbb{Q}]}$. This implies that the number field F_1 is unramified over p.

Let E_1 be a totally imaginary quadratic extension of F_1 which is unramified above p. From [De2, Prop. 2.3.10] we get that there exists an injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that the properties 1.4 (i) and (iv) hold and moreover the torus $Z^0(G)$ is a subtorus of $\mathrm{Res}_{E_1/\mathbb{Q}}\mathbb{G}_m$. From the last two sentences we get that the torus $Z^0(G)$ splits over a Galois extension of $\mathbb{Q}$ unramified above p. Thus the torus $Z^0(G)_{\mathbb{Q}_p}$ is unramified. As the group $G_{\mathbb{Q}_p}$ is isogeneous to $Z^0(G)_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} G_{1,\mathbb{Q}_p}$, we get that it is unramified.

Let H be a hyperspecial subgroup of $G(\mathbb{Q}_p)$. Let $G_{\mathbb{Z}_p}$ be a reductive group scheme over $\mathbb{Z}_p$ whose generic fibre is $G_{\mathbb{Q}_p}$ and for which we have an identity $H = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Let $G_{\mathbb{Z}_{(p)}}$ be the reductive group scheme over $\mathbb{Z}_{(p)}$ whose generic fibre is G and whose extension to $\mathbb{Z}_p$ is $G_{\mathbb{Z}_p}$, cf. proof of [Va1, Lem. 3.1.3].

Let $H_1' := G^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. The hyperspecial subgroups of $G_1(\mathbb{Q}_p)$ are $G_1(\mathbb{Q}_p)$-conjugate, cf. [Ti2, p. 47]. Thus there exists an element $g \in G_1(\mathbb{Q}_p)$ such that we have $\tilde{H}_1 = gH_1g^{-1}$. By replacing H with $g^{-1}Hg$, $\tilde{H}_1$ gets

replaced by $H_1 = g^{-1}\tilde{H}_1 g$. Therefore we can assume that $\tilde{H}_1 = H_1$. This implies that $G_{1,\mathbb{Z}_p} = G^{\mathrm{ad}}_{\mathbb{Z}_p}$. Thus the property 1.4 (iii) holds.

Let L be a $\mathbb{Z}$-lattice of W which is self-dual with respect to ψ (i.e., ψ induces a perfect alternating form $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$). From [Va9, Lem. 4.2.1] we get that we can modify the $\mathbb{Z}$-lattice L and the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that moreover $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a $G_{\mathbb{Z}_{(p)}}$-module. The resulting homomorphism $G_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_{L_{(p)}}$ is a closed embedding, cf. [Va4, Thm. 1.1] and [Va9, Fact 2.3.1]. Thus the property 1.4 (ii) holds as well. $\square$

2.3. Definition. A morphism $q : (G_1, \mathcal{X}_1, H_1) \to (\tilde{G}_1, \tilde{\mathcal{X}}_1, \tilde{H}_1)$ between Shimura triples of abelian type is called a *cover*, if the following two properties hold:

(i) the group G_1 surjects onto $\tilde{G}_1$, and

(ii) the kernel $\mathrm{Ker}(q)$ is a subtorus of $Z(G_1)$ with the property that for every field K of characteristic 0 the group $H^1(K, \mathrm{Ker}(q)_K)$ is trivial.

Each cover $q : (G_1, \mathcal{X}_1, H_1) \to (\tilde{G}_1, \tilde{\mathcal{X}}_1, \tilde{H}_1)$ induces at the level of adjoint triples an isomorphism $(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}) \xrightarrow{\sim} (\tilde{G}_1^{\mathrm{ad}}, \tilde{\mathcal{X}}_1^{\mathrm{ad}}, \tilde{H}_1^{\mathrm{ad}})$.

2.4. Proposition. *Let $(G_1, \mathcal{X}_1, H_1) \hookrightarrow (G_2, \mathcal{X}_2, H_2)$ be an injective map of Shimura triples of abelian type with respect to p. For $j \in \{1, 2\}$ we assume that the integral canonical model of $(G_j, \mathcal{X}_j, H_j)$ exists and is quasi-projective. We view $\mathcal{N}_1$ as a $\mathcal{N}_2$-scheme via the functorial morphism $\mathcal{N}_1 \to \mathcal{N}_2$ of $E(G_2, \mathcal{X}_2)_{(p)}$-schemes. Then $\mathcal{N}_1$ is the normalization $\mathcal{P}_1$ of the Zariski closure of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ in $\mathcal{N}_2$.*

Proof: It is known that $\mathcal{P}_1$ is a normal integral model of $(G_1, \mathcal{X}_1, H_1)$ that has the extension property, cf. [Va1, Prop. 3.4.1]. We have a natural morphism $a : \mathcal{N}_1 \to \mathcal{P}_1$ of $E(G_1, \mathcal{X}_1)_{(p)}$-schemes whose generic fibre is the identity automorphism of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$. The morphism a is a pro-étale cover of a morphism $a_{H_0} : \mathcal{N}_1/H_0 \to \mathcal{P}_1/H_0$ of normal $E(G_1, \mathcal{X}_1)_{(p)}$-schemes of finite type, where H_0 is a small enough compact, open subgroup of $G_1(\mathbb{A}_f^{(p)})$ (cf. Definition 1.3.1 (a)). As $\mathcal{N}_2$ is quasi-projective and as $E(G_1, \mathcal{X}_1)_{(p)}$ is an excellent ring, it is easy to see that the $E(G_1, \mathcal{X}_1)_{(p)}$-scheme $\mathcal{P}_1/H_0$ is quasi-projective. Thus a_{H_0} is a quasi-projective morphism between normal, flat $E(G_1, \mathcal{X}_1)_{(p)}$-schemes of finite type whose generic fibre is an isomorphism. As each discrete valuation ring of mixed characteristic $(0, p)$ is a healthy regular scheme, the morphism a satisfies the valuative criterion of properness with respect to such discrete valuation rings. From the last two sentences we get that a_{H_0} is in fact a projective morphism.

We consider an open subscheme $\mathcal{O}_1$ of $\mathcal{P}_1$ which contains $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ and for which the morphism $a^{-1}(\mathcal{P}_1) \to \mathcal{P}_1$ is an isomorphism. As $\mathcal{N}_1$ has the extension property (cf. Definition 1.3.1 (d)) and as each regular, formally smooth scheme over $E(G_1, \mathcal{X}_1)_{(p)}$ is healthy, we easily get that we can assume that $\mathcal{O}_1$ contains the formally smooth locus of $\mathcal{P}_1$ over $E(G_1, \mathcal{X}_1)_{(p)}$. As a_{H_0} is projective, we can also assume that we have an inequality $\mathrm{codim}_{\mathcal{P}_1}(\mathcal{P}_1 \setminus \mathcal{O}_1) \geq 2$. Obviously we can assume that $\mathcal{O}_1$ is H_0-invariant. Thus the projective morphism $a_{H_0} : \mathcal{N}_1/H_0 \to \mathcal{P}_1/H_0$ is an isomorphism above $\mathcal{O}_1/H_0$.

Let $\mathcal{Y}$ be the set of points of $\mathcal{N}_1/H_0$ which are of codimension 1 and which do not belong to $a_{H_0}^{-1}(\mathcal{O}_1/H_0) \xrightarrow{\sim} \mathcal{O}_1/H_0$. To prove the Proposition it suffices to show that a_{H_0} is

an isomorphism. To check that a_{H_0} is an isomorphism, it suffices to show that the set $\mathcal{Y}$ is empty (this is so as the projective morphism a_{H_0} is a blowing up of a closed subscheme of $\mathcal{P}_1/H_0$; the proof of this is similar to [Ha, Ch. II, Thm. 7.17]).

We show that the assumption that the set $\mathcal{Y}$ is non-empty leads to a contradiction. Let $\mathcal{C}$ be the open subscheme of $\mathcal{N}_1/H_0$ which is the union of: (i) the generic fibre of $\mathcal{N}_1/H_0$ and (ii) the union $\mathcal{E}_1$ of all connected components of the special fibre of $\mathcal{N}_1/H_0$ whose generic points belong to $\mathcal{Y}$. The image $\mathcal{E}_2 := a_{H_0}(\mathcal{E}_1)$ has dimension less than $\mathcal{E}_1$ and is contained in the non-smooth locus of $\mathcal{P}_1/H_0$. The morphism $\mathcal{C} \to \mathcal{P}_1/H_0$ factors through the dilatation $\mathcal{V}$ of $\mathcal{P}_1/H_0$ centered on the reduced scheme of the non-smooth locus of $\mathcal{P}_1/H_0$, cf. the universal property of dilatations (see [BLR, Ch. 3, 3.2, Prop. 3.1 (b)]). But $\mathcal{V}$ is an affine $\mathcal{P}_1/H_0$-scheme and thus the image of the projective $\mathcal{P}_1/H_0$-scheme $\mathcal{E}_1$ in $\mathcal{V}$ has the same dimension as $\mathcal{E}_2$. By repeating the process we get that the image of $\mathcal{E}_1$ in a smoothening $\mathcal{V}_\infty$ of $\mathcal{P}_1/H_0$ obtaining via a sequence of blows up centered on non-smooth loci (see [BLR, Ch. 3, Thm. 3 of 3.1 and Thm. 2 of 3.4]), has dimension $\dim(\mathcal{E}_2)$ and thus it has dimension less than $\mathcal{E}_1$. But each discrete valuation ring of $\mathcal{V}_\infty$ dominates a local ring of $\mathcal{N}_1/H_0$ (as a_{H_0} is a projective morphism) and therefore (due to the existence of the morphism $\mathcal{C} \to \mathcal{V}_\infty$) it is also a local ring of $\mathcal{N}_1/H_0$. As $\mathcal{V}_\infty$ has at least one discrete valuation ring of mixed characteristic $(0, p)$ which is not a local ring of $\mathcal{O}_1/H_0$, we get that this discrete valuation ring is the local ring of a point in $\mathcal{Y}$. Thus the image of $\mathcal{E}_1$ in $\mathcal{V}_\infty$ has the same dimension as $\mathcal{E}_1$. Contradiction. Thus the set $\mathcal{Y}$ is empty and $a : \mathcal{N}_1 \to \mathcal{P}_1$ is an isomorphism. $\square$

3. The proof of Theorem 1.5

Subsections 3.1 to 3.5, 3.7, and 3.8 consider the seven cases needed to prove Theorem 1.5 (i). Subsection 3.6 recalls notations that are required for Subsections 3.7 and 3.8. Theorem 1.5 (ii) and (iii) is proved in Subsection 3.10, based on Proposition 3.9. Corollary 3.11 is the very essence of the proof of Theorem 1.6 (a) and (b).

Let $(G_1, \mathcal{X}_1)$ be a simple, adjoint Shimura pair of abelian type. We assume that the group $G_{1,\mathbb{Q}_p}$ is unramified. Let H_1 be a hyperspecial subgroup of $G_1(\mathbb{Q}_p)$ i.e., the group of $\mathbb{Z}_p$-valued points of a reductive group scheme $G_{1,\mathbb{Z}_p}$ over $\mathbb{Z}_p$ that extends $G_{1,\mathbb{Q}_p}$. Until Subsection 1.3 we will consider an injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{S})$ of Shimura pairs such that the properties 1.4 (i) to (iv) hold.

Let $\mathcal{O} :=:= \mathrm{End}(L_{(p)}) \cap \{e \in \mathrm{End}(W) | e \text{ is fixed by } G_1\}$.

3.1. Case 1: the unitary case. We assume that the Shimura pair $(G_1, \mathcal{X}_1)$ is unitary. Based on [Va6, Prop. 3.2], we can assume that the $\mathbb{Z}_{(p)}$-algebra $\mathcal{O}$ is semisimple and that $G_{\mathbb{Z}_{(p)}}$ is the subgroup scheme of $\mathbf{GSp}(L_{(p)}, \psi)$ that fixes $\mathcal{O}$. From this and [Va6, Cor. 4.1.1] we get that the property 1.5 (i) holds. The fact that the properties 1.5 (ii) and (iii) hold follows from [Va6, Thm. 5.1 (a) and (b) (respectively). Thus Theorem 1.5 holds if the Shimura pair $(G_1, \mathcal{X}_1)$ is unitary.

3.2. Case 2: the totally non-compact C_n type case. We assume that the Shimura pair $(G_1, \mathcal{X}_1)$ is of C_n type and that all simple factors of $G_{1,\mathbb{R}}$ are non-compact. Then arguments entirely similar to the ones of [Va6, Prop. 3.2] show that we can assume that

the $\mathbb{Z}_{(p)}$-algebra $\mathcal{O}$ is semisimple and that $G_{\mathbb{Z}_{(p)}}$ is the subgroup scheme of $\mathbf{GSp}(L_{(p)},\psi)$ that fixes $\mathcal{O}$. From this and [Zi, Subsect. 3.5], [LR], and [Ko, Sect. 5] we get that the property 1.5 (i) holds and in fact $\mathcal{N}$ is the Zariski closure of $\mathrm{Sh}(G,\mathcal{X})/J_p$ in $\mathcal{M}_{E(G,\mathcal{X})_{(p)}}$. Using this, [Va6, Subsections 4.1 to 4.3 and 5.1] can be easily adapted to show that the properties 1.5 (ii) and (iii) hold as well. Thus Theorem 1.5 holds if the Shimura pair $(G_1,\mathcal{X}_1)$ is of C_n type and all simple factors of $G_{1,\mathbb{R}}$ are non-compact.

3.3. Case 3: the totally non-compact $D_n^{\mathbb{H}}$ type case with $p>2$. We assume that $p>2$, that the Shimura pair $(G_1,\mathcal{X}_1)$ is of $D_n^{\mathbb{H}}$ type, and that all simple factors of $G_{1,\mathbb{R}}$ are non-compact. Then arguments entirely similar to the ones of [Va6, Prop. 3.2] show that we can assume that the $\mathbb{Z}_{(p)}$-algebra $\mathcal{O}$ is semisimple and that $G_{\mathbb{Z}_{(p)}}$ is the identity component of the subgroup scheme of $\mathbf{GSp}(L_{(p)},\psi)$ that fixes $\mathcal{O}$. From this and [Ko, Sect. 5] we get that the property 1.5 (i) holds and in fact $\mathcal{N}$ is the Zariski closure of $\mathrm{Sh}(G,\mathcal{X})/J_p$ in $\mathcal{M}_{E(G,\mathcal{X})_{(p)}}$ and is quasi-projective. Using this, [Va6, Subsections 4.1 to 4.3 and 5.1] can be easily adapted to show that the properties 1.5 (ii) and (iii) hold as well. Thus Theorem 1.5 holds if $p>2$, the Shimura pair $(G_1,\mathcal{X}_1)$ is of $D_n^{\mathbb{H}}$ type, and all simple factors of $G_{1,\mathbb{R}}$ are non-compact.

We point out a second way to argue that the properties 1.5 (ii) and (iii) hold. Let $(G',\mathcal{X}',H')\to(G_1,\mathcal{X}_1,H_1)$ be a map of Shimura triples for which the following two properties hold (cf. [MS, 3.4] and [Va1, Rm. 3.2.7 10)]): (i) it is a cover in the sense of Subsection 2.3 and (ii) we have identities $E(G',\mathcal{X}')=E(G_1,\mathcal{X}_1)$ and $G'^{,\mathrm{der}}=G^{\mathrm{der}}$. As $G'^{,\mathrm{der}}=G^{\mathrm{der}}$, the integral canonical model $\mathcal{N}'$ of $(G',\mathcal{X}',H')$ over $E(G',\mathcal{X}')_{(p)}=E(G_1,\mathcal{X}_1)_{(p)}$ exists (cf. [Va6, Prop. 4.2.3 (a)] and the fact that the integral canonical model $\mathcal{N}$ of $(G,\mathcal{X},J_p)$ exists). As $\mathcal{N}$ is quasi-projective, from [Va6, Prop. 2.4.3 (c)] we easily get that $\mathcal{N}'$ is also quasi-projective. The order of the center of the simply connected semisimple group cover of G_1 is a power of 2 and thus it is relative prime to p. From the last two sentences and [Va1, Thm. 6.2 (a)] we get that the integral canonical model $\mathcal{N}_1$ of $(G_1,\mathcal{X}_1,H_1)$ exists and that the functorial morphism $\mathcal{N}'\to\mathcal{N}_1$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_1$. From this and [Va6, Prop. 4.2.3 (c)] we easily get the functorial morphism $\mathcal{N}'\to\mathcal{N}_1$ is as well a pro-étale cover of an open closed subscheme of $\mathcal{N}_1$. Thus the properties 1.5 (ii) and (iii) hold.

3.4. Case 4: the totally non-compact $D_n^{\mathbb{H}}$ type case with $p=2$. We assume that $p=2$, that the Shimura pair $(G_1,\mathcal{X}_1)$ is of $D_n^{\mathbb{H}}$ type, and that all simple factors of $G_{1,\mathbb{R}}$ are non-compact. Then arguments entirely similar to the ones of [Va6, Prop. 3.2] show that we can assume that the $\mathbb{Z}_{(p)}$-algebra $\mathcal{O}$ is semisimple and that $G_{\mathbb{Z}_{(p)}}$ is the Zariski closure in $\mathbf{GSp}(L_{(p)},\psi)$ of the identity component of the subgroup scheme of $\mathbf{GSp}(L_{(p)},\psi)$ that fixes $\mathcal{O}$. From this and [Va8] we get that the property 1.5 (i) holds.

3.5. Case 5: the non-unitary, compact factors case with $p>2$. We assume that $p>2$ and that the Shimura pair $(G_1,\mathcal{X}_1)$ has compact factors and is not unitary. From [Va9, Cor. 1.7 (a)] we get that the property 1.5 (i) holds. As $(G_1,\mathcal{X}_1)$ is not unitary, it is of B_n, C_n, $D_n^{\mathbb{H}}$, or $D_n^{\mathbb{R}}$ type. Thus the order of the center of the simply connected semisimple group cover of G_1 is a power of 2. Thus as in the second paragraph of Section 3.3 we argue that the properties 1.5 (ii) and (iii) hold.

3.6. Notations. In this Subsection we list notations that are required for the last two cases (see Subsections 3.7 and 3.8). Let ψ^* be the perfect alternating form on $L^*_{(p)}$ that is defined naturally by ψ. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of tensors of $\mathcal{T}(W^*)$ such that G is the subgroup of $\mathbf{GL}_{W^*}$ that fixes v_α for all $\alpha\in\mathcal{J}$.

Let $\mathcal{N}$ be defined as in the condition 1.5 (i). Let $\mathcal{N}^{\mathrm{s}}$ be the open subscheme of $\mathcal{N}$ which is the formally smooth locus of $\mathcal{N}$ over $E(G,\mathcal{X})_{(2)}$. We have an identity $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})}=\mathcal{N}_{E(G,\mathcal{X})}$, cf. [Va9, Lem. 2.2.2]. To show that the property 1.5 (i) holds, it suffices to check that $\mathcal{N}^{\mathrm{s}}=\mathcal{N}$ (cf. [Va1, Cor. 3.4.4]). Let $(\mathcal{A},\Lambda_{\mathcal{A}})$ be the pull back to $\mathcal{N}$ of the universal principally polarized abelian scheme over $\mathcal{M}$.

Let k be an algebraically closed field of characteristic p which is countable transcendental degree. Let $W(k)$ be the ring of Witt vectors with coefficients in k. Let $B(k):=W(k)[\frac{1}{p}]$. Let σ be the Frobenius automorphism of k, $W(k)$, and $B(k)$. Let $R_0:=W(k)[[x]]$, where x is an independent variable. Let Φ_0 be the Frobenius lift of R_0 that is compatible with σ and takes x to x^p.

For $z\in\mathcal{N}^{\mathrm{s}}(W(k))$, let $(A,\lambda_A):=z^*(\mathcal{A},\Lambda_{\mathcal{A}})$. Let (M,F^1,ϕ,ψ_M) be the principally quasi-polarized filtered Dieudonné module of the principally quasi-polarized p-divisible group of (A,λ_A). For each $\alpha\in\mathcal{J}$ let t_α be the De Rham component of the Hodge cycle on $A_{B(k)}$ that corresponds naturally to v_α (see [Va9, Subsect. 3.2]). It is known that there exists an open subscheme $\mathcal{N}^{\mathrm{m}}$ of $\mathcal{N}^{\mathrm{s}}$ that contains $\mathcal{N}_{E(G,\mathcal{X})}$ and such that the point $z\in\mathcal{N}(W(k))$ belongs to $\mathcal{N}^{\mathrm{m}}(W(k))$ if and only if there exists an isomorphism

$$\rho_z:(M,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)\xrightarrow{\sim}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}W(k),(v_\alpha)_{\alpha\in\mathcal{J}},\psi^*),$$

cf. [Va9, Subsubsect. 3.5.1] and [Va7, Rm. 4.4 (a)]. We refer to $(M,F^1,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)$ as the *principally quasi-polarized filtered Dieudonné module with tensors attached* to z. If $p>2$, then we have $\mathcal{N}^{\mathrm{m}}=\mathcal{N}^{\mathrm{s}}$, cf. [Va9, Thm. 3.2.2 (a)] and [Va7, Rm. 4.4 (a)].

3.7. Case 6: the non-unitary, compact factors case with $p=2$. We assume that $p=2$ and that the Shimura pair $(G_1,\mathcal{X}_1)$ has compact factors and is not unitary.

The fibres of the morphism $\mathcal{N}^{\mathrm{m}}\to\mathrm{Spec}(E(G,\mathcal{X})_{(2)})$ are non-empty, cf. [Va9, Thm. 3.3.2]. As in the proofs of [Va9, Thms. 1.6 (c) and 1.7], to show that $\mathcal{N}^{\mathrm{m}}=\mathcal{N}^{\mathrm{s}}=\mathcal{N}$, it suffices to show that $\mathcal{N}^{\mathrm{m}}_{\mathbb{F}_2}$ is an open closed subscheme of $\mathcal{N}_{\mathbb{F}_2}$. To check this, we only have to show that for each commutative diagram of the following type

$$\begin{array}{ccccc}\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[[x]]) & \longleftarrow & \mathrm{Spec}(k((x)))\\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_{k((x))}}\\ \mathcal{N} & \longleftarrow & \mathcal{N}_{\mathbb{F}_2} & \longleftarrow & \mathcal{N}^{\mathrm{m}}_{\mathbb{F}_2},\end{array}\tag{8}$$

the morphism $y:\mathrm{Spec}(k)\to\mathcal{N}$ factors through the open subscheme $\mathcal{N}^{\mathrm{m}}$ of $\mathcal{N}$. All the horizontal arrows of the diagram (8) are natural embeddings.

We consider the principally quasi-polarized F-crystal

$$(M_0,\Phi_0,\nabla_0,\psi_{M_0})$$

over $k[[x]]$ of $q^*((\mathcal{A}, \Lambda_{\mathcal{A}}) \times_{\mathcal{N}} \mathcal{N}_{\mathbb{F}_2})$. Thus M_0 is a free R_0-module of rank $2d$, Φ_0 is a Φ_{R_0}-linear endomorphism of M_0, and ∇_0 is an integrable and nilpotent modulo 2 connection on M_0 such that we have $\nabla_0 \circ \Phi_0 = (\Phi_0 \otimes d\Phi_{R_0}) \circ \nabla_0$. Let K_0 be the field of fractions of R_0. In [Va9, Subsect. 5.1 and Thm. 5.2] it is proved that:

(i) to each element $\alpha \in \mathcal{J}$ corresponds a tensor $t_{0,\alpha} \in \mathcal{T}(M_0[\frac{1}{p}])$;

(ii) the Zariski closure in $\mathbf{GL}_{M_0}$ of the subgroup of $\mathbf{GL}_{M_0 \otimes_{R_0} K_0}$ that fixes $t_{0,\alpha}$ for all $\alpha \in \mathcal{J}$, is a reductive group scheme $\mathcal{G}_0$ over $\mathrm{Spec}(R_0)$.

In [Va9, Subsect. 5.3] it is proved that there exists a cocharacter $\mu_0 : \mathbb{G}_m \to \mathcal{G}_0$ that gives birth to a decomposition $M_0 = F_0^1 \oplus F_0^0$ such that $\mathbb{G}_m$ acts through μ_0 on F_0^i via the $-i^{\mathrm{th}}$-power of the identity character of $\mathbb{G}_m$. Moreover, the triple $(M_0, F_0^1, \nabla_0, \psi_{M_0})$ is a principally quasi-polarized filtered F-crystal over $k[[x]]$. Let

$$\mathcal{C}_0 := (M_0, F_0^1, \Phi_0, \nabla_0, (t_{0,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_0}).$$

Let k_1 be an algebraic closure of $k((x))$. Let $z_1 : \mathrm{Spec}(W(k_1)) \to \mathrm{Spec}(R_0)$ be a Teichmüller lift that lifts the natural morphism $\mathrm{Spec}(k_1) \to \mathrm{Spec}(k[[x]])$. The pull back of $\mathcal{C}_0$ to $\mathrm{Spec}(W(k_1))$ is of the form $(M_1, F_1^1, \phi_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_1})$ and (cf. [Va9, Lem. 3.5.2]) is the principally quasi-polarized filtered Dieudonné module with tensors attached to a point $z_1^{\mathrm{m}} \in \mathcal{N}^{\mathrm{m}}(W(k_1))$. The point $y_1 \in \mathcal{N}^{\mathrm{m}}(k_1)$ defined by z_1^{m} is the one defined naturally by the composite of $\mathrm{Spec}(k_1) \to \mathrm{Spec}(k((x)))$ with $q_{k((x))} : \mathrm{Spec}(k((x))) \to \mathcal{N}^{\mathrm{m}}_{\mathbb{F}_2}$. The point $z_1^{\mathrm{m}} \in \mathcal{N}^{\mathrm{m}}(W(k_1))$ defines naturally a structure on $W(k_1)$ as a $E(G, \mathcal{X})_{(2)}$-algebra.

Let $(M, F^1, \Phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$ be the reduction modulo x of $\mathcal{C}_0$ (the connection being ignored, as it is trivial). Let (D, λ_D) be a principally quasi-polarized 2-divisible group over $W(k)$ whose principally quasi-polarized filtered F-crystal is (M, F^1, ϕ, ψ_M) and for which there exists an isomorphism $r_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H^1(D) \otimes_{\mathbb{Z}_2} W(k), (v_\alpha^D)_{\alpha \in \mathcal{J}})$ that takes ψ_M to the perfect alternating form $\lambda_{H^1(D)}$ on $H^1(D)$ defined by λ_D (see Appendix). Here $H^1(D) := T_2(D^{\mathrm{t}}_{B(k)})(-1)$ is the dual of the Tate-module $T_2(D_{B(k)})$ of $D_{B(k)}$ and $(v_\alpha^D)_{\alpha \in \mathcal{J}}$ is the family of étale Tate-cycles on $D_{B(k)}$ that corresponds to $(t_\alpha)_{\alpha \in \mathcal{J}}$ via Fontaine comparison theory for D. Each v_α^D is a tensor of $\mathcal{T}(H^1(D)[\frac{1}{p}])$ fixed by $\mathrm{Gal}(B(k))$. As $z_1 \in \mathcal{N}^{\mathrm{m}}(W(k_1))$, in [Va9, Rm. 5.6.1] it is checked that there exists an isomorphism $r_0 : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$. As in [Va7, Rm. 4.4 (a)] we argue that we can assume that under r_0, ψ_M is mapped to $\lambda_{H^1(D)}$. Based on this and the existence of r_D, we get that there exists an isomorphism

$$l_D : (H^1(D) \otimes_{\mathbb{Z}_2} W(k), (v_\alpha^D)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$$

that takes $\lambda_{H^1(D)}$ to ψ^*.

Let $(\mathcal{D}_0, \lambda_{\mathcal{D}_0})$ be the principally quasi-polarized 2-divisible group over R_0 whose principally quasi-polarized filtered F-crystal is $(M_0, F_0^1, \Phi_0, \nabla_0, \psi_{M_0})$ and whose reduction modulo (x) is (D, λ_D), cf. [Va9, Lem. B7.2]. To it corresponds naturally a morphism $z_0 : \mathrm{Spec}(R_0) \to \mathcal{M}$ whose reduction modulo the ideal $(2, x)$ of R_0 is the k-valued point of $\mathcal{M}$ defined naturally by y. Due to the existence of r_D and l_D and the fact that $\mathcal{N}^{\mathrm{s}}(W(k)) = \mathcal{N}(W(k))$ (cf. [Va9, Thm. 1.5 (a)]), to show that $y : \mathrm{Spec}(k) \to \mathcal{N}$ factors through $\mathcal{N}^{\mathrm{m}}$ it

suffices to show that $z_0 : \mathrm{Spec}(R_0) \to \mathcal{M}$ factors through $\mathcal{N}$ in a way in which it lifts the point $y : \mathrm{Spec}(k) \to \mathcal{N}$. For this it suffices to show that $z_1^{\mathrm{s}} := z_0 \circ z_1 : \mathrm{Spec}(W(k_1)) \to \mathcal{M}$ is the same $W(k)$-valued point of $\mathcal{M}$ as the one defined by a suitable point $z_1^{\mathrm{m}} \in \mathcal{N}^{\mathrm{m}}(W(k_1))$.

Let $(D_1^{\mathrm{m}}, \lambda_{D_1^{\mathrm{m}}})$ and $(D_1^{\mathrm{s}}, \lambda_{D_1^{\mathrm{s}}})$ be the principally quasi-polarized 2-divisible groups of $z_1^{\mathrm{m}*}(\mathcal{A}, \Lambda_{\mathcal{A}})$ and of the pull back via z_1^{s} of the universal principally polarized abelian scheme over $\mathcal{M}$. As the principally quasi-polarized filtered Dieudonné modules of $(D_1^{\mathrm{m}}, \lambda_{D_1^{\mathrm{m}}})$ and $(D_1^{\mathrm{s}}, \lambda_{D_1^{\mathrm{s}}})$ are both equal to $(M_1, F_1^1, \phi_1, \psi_{M_1})$, we can identify $H_1(D_1^{\mathrm{m}})[\frac{1}{2}] = H_1(D_1^{\mathrm{s}})[\frac{1}{2}]$. Based on this and [Va9, Lem. 2.3.4], we can identify $H_1(D_1^{\mathrm{m}}) = L_{(2)}^* \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2$ and $H^1(D_1^{\mathrm{m}})[\frac{1}{2}] = H^1(D_1^{\mathrm{s}})[\frac{1}{2}]$ in such way that ψ^* is the étale realization of both $\lambda_{D_1^{\mathrm{m}}}$ and $\lambda_{D_1^{\mathrm{s}}}$ and moreover each v_α is the tensor $v_\alpha^{D_1^{\mathrm{m}}} = v_\alpha^{D_1^{\mathrm{s}}}$ that corresponds to $t_{1,\alpha}$ via Fontaine's comparison theory for either D_1^{m} or D_1^{s}. As $(D_1^{\mathrm{s}}, \lambda_{D_1^{\mathrm{s}}})$ is a deformation of (D, λ_D), we can identify canonically $(H^1(D_1^{\mathrm{s}}), \lambda_{H^1(D_1^{\mathrm{s}})}) = (H^1(D), \lambda_{H^1(D)})$. It is easy to see that under this identification, each tensor $v_\alpha^{D_1^{\mathrm{s}}}$ gets identifies with v_α^D. Due to this and the existence of l_D, we get that there exists an element $g \in G(\mathbb{Q}_2)$ that fixes ψ^* and such that we have $g(L_{(2)}^* \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2) = H^1(D_1^{\mathrm{s}})$. As $G(\mathbb{Q}_2) = G(\mathbb{Q})G_{\mathbb{Z}_{(2)}}(\mathbb{Z}_2)$ (cf. [Mi3, Lem. 4.9]) and as the analogue of this holds for the connected subgroup C^0 of G which is the stabilizer of ψ^* (or ψ) in G, we can assume that we have $g \in G^0(\mathbb{Q})$.

Let $z_1^{\mathrm{s}} g^{-1}$ be the right translate of z_1^{s} through the element $g^{-1} \in \mathbf{Sp}(W, \psi)(\mathbb{A}_f^{(2)})$. We consider an $E(G, \mathcal{X})_{(2)}$-monomorphism $W(k_1) \hookrightarrow \mathbb{C}$. Via it, we can view both z_1^{s} and z_1^{m} as $\mathbb{C}$-valued points of $\mathcal{M}$. We lift them to $\mathbb{C}$-valued points $z_{1,\infty}^{\mathrm{s}}$ and $z_{1,\infty}^{\mathrm{m}}$ of $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})$. To fix the notations, we can assume that the $\mathbb{C}$-valued point

$$z_{1,\infty}^{\mathrm{m}} \in \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})(\mathbb{C}) = \mathbf{GSp}(W, \psi)(\mathbb{Q}) \backslash (\mathcal{S} \times \mathbf{GSp}(W, \psi)(\mathbb{A}_f))$$

is defined by the equivalence class $[x_1, 1_W]$ for some element $x_1 \in \mathcal{X}_1 \subseteq \mathcal{S}$ and for the identity element 1_W of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f)$. The fact that we have an identity $g(L_{(2)}^* \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2) = H^1(D_1^{\mathrm{s}})$, means that we can choose the $\mathbb{C}$-valued point $z_{1,\infty}^{\mathrm{s}}$ in such a way that it is defined by the equivalence class $[x_1, g]$, where g is viewed here only as an element of $\mathbf{Sp}(W, \psi)(\mathbb{Q}_2)$. From the last two sentences we get that z_1^{m} and $z_1^{\mathrm{s}} g^{-1}$ define the same $W(k_1)$-valued point of $\mathcal{M}$. Based on this and the fact that the right translation by g^{-1} is an automorphism of $\mathcal{M}_{E(G,\mathcal{X})_{(2)}}$ that normalizes its locally closed subscheme $\mathcal{N}_{E(G,\mathcal{X})}$, we get that $z_{1,B(k)}^{\mathrm{s}}$ factors naturally through $\mathcal{N}_{E(G,\mathcal{X})}$ and thus that $z_1^{\mathrm{s}} \in \mathcal{N}(W(k_1))$. This means that we can choose $z_1^{\mathrm{m}} \in \mathcal{N}^{\mathrm{m}}(W(k_1))$ to be z_1^{s} itself. We conclude that z_0 factors through $\mathcal{N}$ in such a way that it lifts the k-valued point of $\mathcal{N}$. This ends the argument that $\mathcal{N}_{\mathbb{F}_2}^{\mathrm{m}}$ is an open closed subscheme of $\mathcal{N}_{\mathbb{F}_2}$ and thus that $\mathcal{N}^{\mathrm{m}} = \mathcal{N}^{\mathrm{s}} = \mathcal{N}$. Thus the property 1.5 (i) holds.

3.8. Case 7: the totally non-compact B_n and $D_n^{\mathbb{R}}$ types case. We assume that the Shimura pair $(G_1, \mathcal{X}_1)$ is of either B_n or $D_n^{\mathbb{R}}$ type and that all simple factors of $G_{1,\mathbb{R}}$ are non-compact. We have $E(G_1, \mathcal{X}_1) = \mathbb{Q}$, cf. [De2, Rm. 2.3.12]. We can assume that $Z^0(G) = \mathbb{G}_m$ and $E(G, \mathcal{X}) = \mathbb{Q}$, cf. [De2, Rm. 2.3.13]. This implies that there exists a cocharacter $\mu : \mathbb{G}_m \to G_{\mathbb{Z}_p}$ whose extension to $\mathbb{C}$ is $G(\mathbb{C})$-conjugate to the Hodge cocharacters $\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$ with $x \in \mathcal{X}$. We define a positive integer m via the following two rules: (i) if $(G_1, \mathcal{X}_1)$ is of B_n type, then $m := 2n+1$ and (ii) if $(G_1, \mathcal{X}_1)$ is of $D_n^{\mathbb{R}}$ type,

then $m := 2n$. Due to the property 1.4 (iv), the derived group G^{der} is simply connected (cf. also [De2, Table 2.3.8]). Even more, we can also assume that the representation of $G^{\mathrm{der}}_{\mathbb{C}}$ on $W \otimes_{\mathbb{Q}} \mathbb{C}$ is a direct sum of spin representations of direct factors of $G^{\mathrm{der}}_{\mathbb{C}}$ which are $\mathbf{Spin}_m$ groups (see loc. cit. and the proof of [De2, Prop. 2.3.10]). This implies that there exists an epimorphism $\pi : G_{\mathbb{Z}_p} \twoheadrightarrow J_{\mathbb{Z}_p}$ such that the following two properties hold:

(a) The kernel of this epimorphism is a closed, flat subgroup scheme of the center of $G_{\mathbb{Z}_p}$ (and therefore π induces an isomorphism $G^{\mathrm{ad}}_{\mathbb{Z}_p} \xrightarrow{\sim} J^{\mathrm{ad}}_{\mathbb{Z}_p}$ at the level of adjoint group schemes; if $m = 2n+1$, then in fact we have $J_{\mathbb{Z}_p} = G^{\mathrm{ad}}_{\mathbb{Z}_p}$).

(b) The group scheme $J_{\mathbb{Z}_p}$ is a finite product $\prod_{i\in I} J_{i,\mathbb{Z}_p}$ of Weil restrictions of $\mathbf{SO}_m$ group schemes.

Let $\rho_i : J_{i,\mathbb{Z}_p} \to \mathbf{SO}(Q_i, q_i)$ be the corresponding natural special orthogonal representation over $\mathbb{Z}_p$, cf. (b). Here q_i is a perfect quadratic form on a free $W(k_i)$-module Q_i of finite rank and $\mathbf{SO}(Q_i, q_i)$ is viewed as a semisimple group scheme over $\mathbb{Z}_p$, where k_i is a finite field extension of $\mathbb{F}_p$. Let $(Q, b) := \oplus_{i\in I}(Q_i, q_i)$. Let $\rho : J_{\mathbb{Z}_p} \to \mathbf{SO}(Q, b)$ be the direct sum of the special orthogonal representations ρ_i with $i \in I$.

Let $(u_\alpha)_{\alpha\in\mathcal{J}_O}$ be a family of tensors of $\mathcal{T}(Q[\frac{1}{p}])$ such that the generic fibre $\mathbf{SO}_{\mathbb{Q}_p}$ of $\mathbf{SO}_{\mathbb{Z}_p}$ is the subgroup of $\mathbf{GL}_{Q[\frac{1}{p}]}$ that fixes u_α for all $\alpha \in \mathcal{J}_O$. Due to the above description of the representation of $G^{\mathrm{der}}_{\mathbb{C}}$ on $W \otimes_{\mathbb{Q}} \mathbb{C}$, it is easy to see that:

(c) We can view $Q[\frac{1}{p}]$ as a $\mathbb{Q}_p$-submodule of $\mathrm{End}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p) \subseteq \mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

In other words, the standard m-dimensional representation of an $\mathfrak{so}_m$ Lie algebra over $\mathbb{C}$ (or $\overline{\mathbb{Q}_p}$) is a direct summand of the tensor product of the spin representation of $\mathfrak{so}_m$ and of its dual representation.

Due to the property (c), we can view each tensor u_α as a linear combination with coefficients in $\mathbb{Q}_p$ of the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of tensors of $\mathcal{T}(W^*)$ and thus of $\mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. Let $\mu_i : \mathbb{G}_m \to J_{i,\mathbb{Z}_p}$ be the cocharacter induced naturally by μ via the epimorphism $\pi : G_{\mathbb{Z}_p} \twoheadrightarrow J_{\mathbb{Z}_p}$. As Q_i is a $W(k_i)$-module, we can view $W(k_i)$ as a $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(Q_i)$ and thus as a family of tensors of $\mathcal{T}(Q_i)$. If $p > 2$, let b_i be the perfect bilinear form on Q_i defined by q_i; we can view it naturally as a tensor of $Q_i^* \otimes_{\mathbb{Z}_p} Q_i^*$. We have an extra property:

(d) We assume that $p > 2$. Then the family of tensors of $\mathcal{T}(Q_i)$ formed by putting together b_i and $W(k_i)$, is strongly $\mathbb{Z}_p$-very well position for $J_{\mathbb{Z}_p}$ in the sense of [Va1, Def. 4.3.4 and Rm. 4.3.7 1)].

In other words, if C is a flat, reduced $\mathbb{Z}_p$-algebra and if $\tilde{Q}_i$ is a free C-module such that (i) we have $\tilde{Q}_i[\frac{1}{p}] = Q_i \otimes_{\mathbb{Z}_p} C[\frac{1}{p}]$, (ii) b_i induces a perfect bilinear form on $\tilde{Q}_i$, and (iii) we can view naturally $W(k_i) \otimes_{\mathbb{Z}_p} C$ as a C-subalgebra of $\mathrm{End}(\tilde{Q}_i)$, then the Zariski closure of $J_{i,\mathbb{Z}_p} \times_{\mathbb{Z}_p} C[\frac{1}{p}]$ in $\boldsymbol{GL}_{\tilde{Q}_i}$ is a reductive group scheme. To check this, we can assume that the C-algebra $W(k_i) \otimes_{\mathbb{Z}_p} C$ is isomorphic to $C^{[k_i:\mathbb{F}_p]}$ and this case is standard.

In what follows we appeal to the notations of Subsection 3.6. Let n be the dimension of $\mathcal{X}_1$ as a complex manifold (i.e., the dimension of the Shimura variety $\mathrm{Sh}(G_1, \mathcal{X}_1)$).

3.8.1. Proposition. *Let $y \in \mathcal{N}(k)$ be such that the abelian variety $y^*(\mathcal{A})$ is ordinary. Then the point y belongs to $\mathcal{N}^{\mathrm{s}}(k)$. If moreover $p = 2$, then the point y belongs to $\mathcal{N}^{\mathrm{m}}(k)$.*

Proof: The first part is a particular case of [No, Cor. 3.8]. To check the second part we can assume that $p = 2$. Either from loc. cit. or from [Va9, Lem. 3.5.2] we get that there exists a lift $z \in \mathcal{N}^{\mathrm{s}}(W(k))$ of y such that the principally quasi-polarized filtered Dieudonné module with tensors attached to z is of the form $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$, where the Hodge filtration F^1 of M is left invariant by ϕ i.e., we have $\phi(F^1) = 2F^1$. In other words, F^1 is the canonical lift of the ordinary Dieudonné module (M, ϕ).

The 2-adic Galois representation attached to the principally quasi-polarized 2-divisible group (D, λ_D) of $(A, \lambda_A) = z^*(\mathcal{A}, \Lambda_{\mathcal{A}})$ can be identified with a homomorphism

$$\eta_A : \mathrm{Gal}(B(k)) \to \mathbf{GSp}(L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2, \psi^*)(\mathbb{Z}_2)$$

that factors through the maximal compact subgroup of the group of $\mathbb{Q}_2$-valued points of a rank 1 split torus T_0 of $G_{\mathbb{Q}_2}$. Here we are using the natural identification of $H^1(D) = H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_2)$ with $L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2$ and of its principal quasi-polarization defined by λ_D (equivalently by λ_A) with the perfect alternating from ψ^* on $L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2$ defined by ψ (see [Va9, Lem. 2.3.4 (a)]). Let $T_{1,\mathbb{Z}_2}$ a torus of $G_{\mathbb{Z}_2}$ whose generic fibre T_1 is $G(\mathbb{Q}_2)$-conjugate to T_0. Let G^0 be the connected, normal, reductive subgroup of G that fixes ψ. Its Zariski closure in $G_{\mathbb{Z}_{(2)}}$ is a reductive group scheme $G^0_{\mathbb{Z}_{(2)}}$ over $\mathbb{Z}_{(2)}$. We have a short exact sequence $1 \to G^0_{\mathbb{Z}_2} \to G_{\mathbb{Z}_2} \to \mathbb{G}_m \to 1$ and the cocharacter $\mu : \mathbb{G}_m \to G_{\mathbb{Z}_2}$ is a splitting of it. Due to this we easily get that T_0 and T_1 are in fact $G^0(\mathbb{Q}_2)$-conjugate. Let the element $g \in G^0(\mathbb{Q}_2)$ be such that we have $gT_1g^{-1} = T_0$. As we have $G^0_{\mathbb{Z}_{(2)}}(\mathbb{Q}_2) = G^0_{\mathbb{Z}_{(2)}}(\mathbb{Q})G^0_{\mathbb{Z}_{(2)}}(\mathbb{Z}_2)$ (cf. [Mi3, Lem. 4.9]), up to a replacement of $T_{1,\mathbb{Z}_2}$ by a $G^0_{\mathbb{Z}_{(2)}}(\mathbb{Z}_2)$-conjugate of it, we can assume that $g \in G^0(\mathbb{Q})$.

Let $L_1 := g(L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2)$. It is a $\mathbb{Z}_2$-lattice of $H^1(D)[\frac{1}{2}] = L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Q}_2 = W^* \otimes_{\mathbb{Q}} \mathbb{Q}_2$ which is self dual with respect to ψ^*, which is normalized by $\mathrm{Im}(\eta_A)$, and for which the Zariski closure of T_0 in $\mathbf{GL}_{\tilde{L}_2}$ is a rank 1 split torus $T_{0,\mathbb{Z}_p}$. Let (A_1, λ_{A_1}) be the principally quasi-polarized abelian scheme over $W(k)$ which is $\mathbb{Z}[\frac{1}{2}]$-isogeneous to (A, λ_A) and whose 2-divisible group D_1 is canonically defined under the mentioned isogeny by the $\mathbb{Z}_2$-lattice L_1 of $H^1(D)[\frac{1}{2}] = L^*_{(2)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Q}_2$; thus we have $H^1(D_1) = L_1$. The 2-adic Galois representation attached to the principally quasi-polarized 2-divisible group (D_1, λ_{D_1}) of (A_1, λ_{A_1}) is the homomorphism

$$\eta_{A_1} : \mathrm{Gal}(B(k)) \to T_{0,\mathbb{Z}_p}(\mathbb{Z}_p) \hookrightarrow \mathbf{GSp}(L_1, \psi^*)(\mathbb{Z}_2)$$

induced naturally by η_A. We easily get that A_1 is the canonical lift of the ordinary abelian variety $A_{1,k}$. If $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{M}$ is the morphism defined naturally by (A_1, λ_{A_1}) and its prime to 2 symplectic similitude structures induced naturally from those of $(A, \lambda_A) = z^*(\mathcal{A}, \Lambda_{\mathcal{A}})$, then its right translate through the element $g^{-1} \in G(\mathbb{A}_f^{(2)}) \leqslant \mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(2)})$ is the composite of $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ with the natural morphism $\mathcal{N} \to \mathcal{M}$.

As $\mathcal{N}_{E(G,\mathcal{X})}$ is a closed subscheme of $\mathcal{M}_{E(G,\mathcal{X})}$, from the last sentence we get that $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{M}$ factors naturally through a morphism $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{N}$. As A_1 is a canonical lift, it is trivial to check that in fact we have $z_1 \in \mathcal{N}^{\mathrm{m}}(W(k))$ (this result is

also a very particular result of [Va7, Thm. 1.2] applied to the torus $T_{0,\mathbb{Z}_p}$ of $\mathbf{GL}_{H^1(D_1)}$). As $\mathcal{N}^{\mathrm{m}}$ is $G(\mathbb{A}_f^{(2)})$-invariant (cf. [Va9, Subsubsect. 3.5.1]), we get that $z \in \mathcal{N}^m(W(k))$. Thus $y \in \mathcal{N}^{\mathrm{m}}(k)$. $\square$

3.8.2. Extra notations. By enlarging the algebraically closed field k, we can assume that there exists a finite, discrete valuation ring extension V of $W(k)$ such that the normalization $\mathcal{P}$ of $\mathcal{N}_V$ is smooth in characteristic 0 and in codimension 1 (cf. [PY, Appendix]). Let K be the field of fractions of V. Let $\mathcal{O}$ be the open subscheme of $\mathcal{P}_k$ which is the ordinary locus; a point $y \in \mathcal{P}(k)$ belongs to $\mathcal{O}(k)$ if and only if $y^*(\mathcal{A}_{\mathcal{N}_V})$ is an ordinary abelian variety.

3.8.3. Proposition. *The ordinary locus $\mathcal{O}$ is Zariski dense in $\mathcal{P}_k$.*

Proof: To prove this Proposition, we can replace $\mathcal{P}$ by an affine, quasi-finite scheme $\mathcal{P}'$ over $\mathcal{P}$ which is formally smooth over V and whose special fibre $\mathcal{P}'_k$ dominates $\mathcal{P}_k$. Let $\mathcal{O}'$ be the reduced scheme of $\mathcal{P}' \times_{\mathcal{P}} \mathcal{O}$. To prove the Proposition it suffices to show that if $Y = \mathrm{Spec}(R)$ is an affine smooth, connected k-subscheme of $\mathcal{P}'_k$ of dimension n, then the abelian scheme $\mathcal{A}_Y$ is generically ordinary. To check this we can perform the following two types of operations:

(o1) Shrinking Y (i.e., the replacement of Y by an affine, open, Zariski dense subscheme of it);

(o2) Replacement of the pair $(\mathcal{P}, Y)$ by a pair of the form $(\mathcal{P}', Y \times_{\mathcal{P}} \mathcal{P}')$.

Let $\mathcal{R}$ be a smooth, affine $W(k)$-scheme whose reduction modulo p is Y. Let $\mathcal{R}^\wedge$ be the p-adic completion of $\mathcal{R}$ and let $\Phi_{\mathcal{R}}$ be a Frobenius lift of it compatible with σ. We can assume that $Z := \mathrm{Spec}(V \otimes_{W(k)} \mathcal{R}^\wedge)$ is an affine, open closed scheme of the affine scheme defined naturally by the p-adic completion of $\mathcal{P}'$ which lifts Y.

Let $\mathcal{C}$ be the F-isocrystal over R of the p-divisible group of $\mathcal{A}_Y$. It is defined naturally by a projective $\mathcal{R}^\wedge[\frac{1}{p}]$-module $\mathcal{V}$ equipped with a $\mathcal{R}^\wedge[\frac{1}{p}]$-linear isomorphism $\mathcal{V} \otimes_{\mathcal{R}^\wedge[\frac{1}{p}]} {}_{\Phi_{\mathcal{R}}} \mathcal{R}^\wedge[\frac{1}{p}]$. The $K \otimes_{B(k)} \mathcal{R}^\wedge$-module $K \otimes_{B(k)} \mathcal{V}$ is equipped with a direct summand $\mathcal{F}$ defined by the Hodge filtration of the p-divisible group of $\mathcal{A}_Z$. In what follows we will express this property by saying that $\mathcal{C}$ is equipped with a filtration after tensorization over $B(k)$ with K. Accordingly, all the F-isocrystals over R will be equipped with a similar filtration after tensorization over $B(k)$ with K and all morphisms of F-isocrystals over R will be compatible with the corresponding filtrations. Moreover all the Kodaira–Spencer maps of such F-isocrystals over R, will be computed after tensorization over $B(k)$ with K and will be with respect to the mentioned filtrations one gets after tensorization over $B(k)$ with K.

Let κ be the field of fractions of R. Let O be the p-adic completion of the local ring of $\mathcal{R}$ (or of $\mathcal{R}^\wedge$) whose residue field is κ. It is a discrete valuation ring of mixed characteristic $(0, p)$ and index of ramification 1. Let $V_1 := V \otimes_{W(k)} O$ and let K_1 be the field of fractions of V_1. As $Z := \mathrm{Spec}(V \otimes_{W(k)} \mathcal{R}^\wedge)$ is an affine, open scheme of the scheme defined by the p-adic completion of $\mathcal{P}'$ which lifts Y, one has a natural composite morphism $z_1 : \mathrm{Spec}(V \otimes_{W(k)} O) \to \mathcal{P}' \to \mathcal{N}$. Let $\rho_1 : \mathrm{Gal}(K_1) \to G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ be the homomorphism associated naturally to the p-adic Galois representation of $A_1 = z_1^*(\mathcal{A})$. By composing ρ_1 with the natural homomorphism $\pi(\mathbb{Z}_2) : G_{\mathbb{Z}_p}(\mathbb{Z}_p) \to J_{\mathbb{Z}_p}(\mathbb{Z}_p)$, we get a homomorphism $\rho_1^{SO} : \mathrm{Gal}(K_1) \to J_{\mathbb{Z}_p}(\mathbb{Z}_p)$.

Let $\mathcal{I} = \oplus_{i \in I} \mathcal{I}_i$ be the F-subisocrystal of $\mathcal{T}(\mathcal{C}) := \oplus_{s,t \in \mathbb{N} \cup \{0\}} \mathcal{C}^{\otimes s} \otimes \mathcal{C}^{* \otimes t}$ (equivalently, of $\mathrm{End}(\mathcal{C})$) that corresponds naturally to the $G_{\mathbb{Q}_p}$-submodule $Q[\frac{1}{p}] = \oplus_{i \in I} Q_i[\frac{1}{p}]$ of $\mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ (equivalently, of $\mathrm{End}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p)$). By performing the operations (o1) and (o2), we can assume that each $\mathcal{I}_i$ is the F-isocrystal of a latticed F-crystal $\mathcal{Q}_i$ over R which generically (i.e., over κ) corresponds naturally (see the explanations below) to the homomorphism $\rho_{1,i}^{SO} : \mathrm{Gal}(K_1) \to J_{i,\mathbb{Z}_p}(\mathbb{Z}_p)$ defined naturally by ρ_1^{SO}.

We will show that the assumption that the abelian scheme $\mathcal{A}_Y$ is not generically ordinary, leads to a contradiction. By performing the operations (o1) and (o2) we can assume that for each element $i \in I$ the following three properties hold:

(a) $\mathcal{Q}_i$ is equipped naturally with a perfect quadratic form $\mathcal{K}_i$ (that corresponds to q_i via Fontaine's comparison theory).

(b) We can identify $(\mathcal{Q}_i, \mathcal{K}_i)$ with

$$\mathcal{E}_i := (Q_i \otimes_{\mathbb{Z}_p} \mathcal{R}^\wedge, g_{i,\mathcal{R}}(\mu_i(\frac{1}{p}) \otimes \Phi_{\mathcal{R}}), q_i, \nabla_i)$$

for some element $g_{i,\mathcal{R}} \in J_{i,\mathbb{Z}_p}(\mathbb{Z}_p)$ and some integrable, nilpotent modulo p connection ∇_i on $Q_i \otimes_{\mathbb{Z}_p} \mathcal{R}^\wedge$. Under this identification, each tensor $u_\alpha \in \mathcal{T}(Q_i \otimes_{\mathbb{Z}_p} \mathcal{R}^\wedge[\frac{1}{p}])$ generates the F-isocrystal over R that corresponds naturally to the tensor $u_\alpha \in \mathcal{T}(Q_i[\frac{1}{p}])$.

(c) The resulting filtration of $Q_i \otimes_{\mathbb{Z}_p} K \otimes_{B(k)} \mathcal{R}^\wedge[\frac{1}{p}]$ (induced via (b) from the one with which $\mathcal{Q}_i$ is equipped after tensorization over $B(k)$ with K) is such that it induces a filtration of $Q_i \otimes_{\mathbb{Z}_p} V \otimes_{W(k)} \mathcal{R}^\wedge$ defined by a cocharacter of $J_{i,V \otimes_{W(k)} \mathcal{R}^\wedge}$ that lifts the cocharacter $\mu_i : \mathbb{G}_m \to J_{i,\mathbb{Z}_p}$.

This is so as these three properties hold over κ. More precisely:

(i) For $p \geq 5$, the fact that we can assume that the properties (a) to (c) hold over κ follows from [Fa, Thm. 5 iii)] applied over the discrete valuation ring $O_2 := V \otimes_{W(k)} W(\kappa^{\mathrm{perf}})$ that dominates O_1 (based on the property 3.8 (d), the arguments for these are entirely the same as the ones of [Va1, Subsect. 5.2]).

(ii) For $p \geq 2$, one can construct $(\mathcal{Q}_i = \mathcal{Q}_i^+ \oplus \mathcal{Q}_i^0 \oplus \mathcal{Q}_i^-, \mathcal{K}_i)$ directly as follows. The quadratic form $\mathcal{K}_i$ of $\mathcal{I}_i$ is well defined, cf. Fontaine's comparison theory. Based on the property 3.8 (d), one first construct a direct summand $\mathcal{Q}_i^+$ of $\mathcal{Q}_i$ that corresponds to positive Newton polygon slopes and that is a direct summand of the End object (defined by $\mathrm{End}(\mathcal{V})$ of the F-crystal over R of the p-divisible group of A_Y. Second, one constructs $\mathcal{Q}_i^-$ in such a way that the quadratic form on $\mathcal{Q}_i^+ \oplus \mathcal{Q}_i^-$ induced by $\mathcal{K}_i$ is perfect. Third, one chooses $\mathcal{Q}_i^0$ such that it is left invariant by the Frobenius endomorphism of $\mathcal{I}_i$ (one can initially work over $\overline{\kappa}$ and then, up to the operations (o1) and (o2), one gets that in fact $\mathcal{Q}_i^0$ is well defined over κ). The fact that we can choose $\mathcal{Q}_i$ such that $\mathcal{K}_i$ defines a perfect quadratic form on it is implied by the fact that $\mathcal{Q}_i^0$ over $\overline{\kappa}$ corresponds to a split $\mathbf{SO}$ group scheme $\mathcal{H}_i$ over $W(k_i)$. More precisely, the generic fibre $\mathcal{H}_{i,B(k_i)}$ corresponds to an inner form of a direct factor of the derived group of the centralizer of a cocharacter of $\mathbf{Spin}_m$ over $B(k_i)$. Thus $\mathcal{H}_{i,B(k_i)}$ is an inner form of a split, simply connected semisimple group over $B(k_i)$ and therefore (due to [Kn, Thm. 1]) it is the trivial inner form.

Let $\mathcal{R}_0$ be the completion of $\mathcal{R}^\wedge$ at some k-valued point of it. Let Φ_0 be the Frobenius lift of $\mathcal{R}_0$ induced naturally by $\Phi_{\mathcal{R}}$. We can assume that $\mathcal{R}_0 = W(k)[[x_1, \ldots, x_n]]$ and that Φ_0 takes x_j to x_j^p for all $j \in \{1, \ldots, n\}$. Let $g_{i,0} \in J_{i,\mathbb{Z}_p}(W(k))$ be the reduction modulo $(x_1, \ldots, x_n)$ of the element $J_{i,\mathbb{Z}_p}(\mathcal{R}_0)$ defined naturally by $g_{i,\mathcal{R}}$. Let $g_0 := (g_{i,0})_{i \in I} \in J_{\mathbb{Z}_p}(W(k) = \prod_{i \in I} J_{i,\mathbb{Z}_p}(W(k))$.

Let U_0 be the maximal, unipotent, smooth, closed subgroup of $J_{\mathbb{Z}_p}$ with the property that μ acts identically on $\mathrm{Lie}(U_0)$. We can identify $\mathcal{R}_0$ with the local ring of the completion of U_0 along its identity section. Thus we have a universal element $u_0 \in U_0(\mathcal{R}_0)$. Let

$$\mathcal{E}_0 := (Q \otimes_{\mathbb{Z}_p} \mathcal{R}_0, u_0(g_0 \mu(\frac{1}{p}) \otimes \Phi_0), b, \nabla_0),$$

where ∇_0 is an integrable, nilpotent modulo p connection on $Q \otimes_{\mathbb{Z}_p} \mathcal{R}_0$ which is uniquely determined by the identity $\nabla_0 \circ (u_0(\mu(\frac{1}{p}) \otimes \Phi_0)) = (u_0(\mu(\frac{1}{p}) \otimes \Phi_0) \otimes d\Phi_{\mathcal{R}_0}) \circ \nabla_0$. The existence and the uniqueness of ∇_0 are proved as in [Fa, Thm. 10] (if needed, the reader could get this by first working in the context of **Spin** group schemes provided by the representation $G_{\mathbb{Z}_p} \hookrightarrow \mathbf{GL}(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ and then by passing naturally to **SO** group schemes via the epimorphism $\pi : G_{\mathbb{Z}_p} \twoheadrightarrow J_{\mathbb{Z}_p}$). Due to the shape of Φ_0, the connection ∇_0 is congruent modulo $(x_1, \ldots, x_n)$ to $\delta_0 - u_0^{-1} du_0$, where δ_0 is the flat connection on $Q \otimes_{\mathbb{Z}_p} \mathcal{R}_0$ that annihilates $Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p$. From this and the definition of U_0 we get that the connection ∇_0 is versal.

As in [Fa, Sect. 7] one checks that the pull back of $\oplus_{i \in I} \mathcal{E}_i$ to $\mathrm{Spec}(\mathcal{R}_0/p\mathcal{R}_0)$ is the pull back of $\mathcal{E}_0$ via a unique morphism $q_0 : \mathrm{Spec}(\mathcal{R}_0/p\mathcal{R}_0) \to \mathrm{Spec}(\mathcal{R}_0/p\mathcal{R}_0)$ (we repeat that the resulting identification preserves the filtrations). We show that the assumption that this morphism is not dominant leads to a contradiction. From this assumption and the fact that the connection ∇_0 is versal we get that there exists an element $i \in I$ such that the Kodaira–Spencer map of ∇_i has an image whose rank is less than the relative dimension of $U_{i,0} := U_0 \cap J_{i,\mathbb{Z}_p}$. But the Kodaira–Spencer map of the pull back $\mathcal{C}_{\mathcal{R}_0/p\mathcal{R}_0}$ of $\mathcal{C}$ to $\mathrm{Spec}(\mathcal{R}_0/p\mathcal{R}_0)$ can be identified with the tensorization with K over $B(k)$ of the direct sum of the Kodaira–Spencer maps of ∇_i's with $i \in I$ (this is so as the adjoint representation of $G_{\mathbb{Z}_p}$ is the composite of the homomorphism $\pi : G_{\mathbb{Z}_p} \twoheadrightarrow J_{\mathbb{Z}_p}$ with the adjoint representation of $J_{\mathbb{Z}_p}$). From the last two sentences we get that the Kodaira–Spencer map of $\mathcal{C}_{\mathcal{R}_0/p\mathcal{R}_0}$ is not injective. As Y is a quasi-finite over a pro-étale cover of a scheme which is finite over $\mathcal{A}_{d,1,N,k}$, the Kodaira–Spencer map of $\mathcal{C}_{\mathcal{R}_0/p\mathcal{R}_0}$ is injective. Contradiction.

Therefore the morphism q_0 is dominant. It is easy to see that $\mathcal{E}_0$ is generically ordinary (this is a very particular case of [Va13]). From the last two sentences we get that, by performing the operation (o1), we can assume that $\mathcal{I}$ is an ordinary F-isocrystal and therefore that $\mathcal{C}$ is ordinary. This implies that Y is contained in $\mathcal{O}$ and thus that $\mathcal{A}_Y$ is generically ordinary. This contradicts our assumption that the abelian scheme $\mathcal{A}_Y$ is not generically ordinary. Thus the Proposition holds. $\square$

3.8.4. Conclusion. Based on Proposition 3.8.3, we get that the ordinary locus of $\mathcal{N}_{\mathbb{F}_p}$ is Zariski dense in $\mathcal{N}_{\mathbb{F}_p}$. But the ordinary locus of $\mathcal{N}_{\mathbb{F}_p}$ is contained in $\mathcal{N}^{\mathrm{s}}_{\mathbb{F}_p}$ if $p > 2$ and in $\mathcal{N}^{\mathrm{m}}_{\mathbb{F}_p}$ if $p = 2$, cf. Proposition 3.8.1. If $p > 2$, then from [Va8, Thm. 1.6 (a)] we get that the open, Zariski dense subscheme $\mathcal{N}^{\mathrm{s}}_{\mathbb{F}_p}$ of $\mathcal{N}_{\mathbb{F}_p}$ is $\mathcal{N}_{\mathbb{F}_p}$ itself. If $p = 2$, then $\mathcal{N}^{\mathrm{m}}_{\mathbb{F}_p}$ is Zariski

dense in $\mathcal{N}_{\mathbb{F}_p}$ and as in Subsection 3.7 we check that $\mathcal{N}^{\mathrm{m}}_{\mathbb{F}_p}$ is an open closed subscheme of $\mathcal{N}_{\mathbb{F}_p}$. Thus for $p=2$ we have $\mathcal{N}^{\mathrm{m}}_{\mathbb{F}_p}=\mathcal{N}_{\mathbb{F}_p}$ and therefore $\mathcal{N}^{\mathrm{m}}=\mathcal{N}^{\mathrm{s}}=\mathcal{N}$. Thus, regardless of what p is, the property 1.5 (i) holds in this last Case 7.

3.8.5. Remark. If $p\geq 5$, then each $(\mathcal{Q}_i,\mathcal{K}_i)$ is defined globally on $\mathcal{P}_k$ and in fact corresponds canonically to (Q_i,q_i) via Fontaine comparison theory (see [Fa, Thm. 5*]).

3.9. Proposition. *Suppose that $(G_1,\mathcal{X}_1,H_1)$ is a Shimura triple of abelian type with respect to p such that the Shimura pair $(G_1,\mathcal{X}_1)$ is simple, adjoint. Then there exists a commutative diagram of Shimura triples of abelian type*

$$\begin{array}{ccc}(G_4,\mathcal{X}_4,H_4) & \xrightarrow{\pi_2} & (G_3,\mathcal{X}_3,H_3)\\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle q_3}\\ (G_1,\mathcal{X}_1,H_1) & \xrightarrow{q_1} & (G_2,\mathcal{X}_2,H_2)\end{array}$$

such that the following four properties hold:

(i) *the Shimura pair $(G_2,\mathcal{X}_2)$ is adjoint and unitary;*

(ii) *both horizontal maps q_1 and π_2 are injective;*

(iii) *both vertical maps π_1 and q_3 induce isomorphisms at the level of adjoint Shimura triples (i.e., they give birth naturally to isomorphisms $(G_4^{\mathrm{ad}},\mathcal{X}_4^{\mathrm{ad}},H_4^{\mathrm{ad}})\xrightarrow{\sim}(G_1,\mathcal{X}_1,H_1)$ and $(G_3^{\mathrm{ad}},\mathcal{X}_3^{\mathrm{ad}},H_3^{\mathrm{ad}})\xrightarrow{\sim}(G_2^{\mathrm{ad}},\mathcal{X}_2^{\mathrm{ad}},H_2^{\mathrm{ad}})$;*

(iv) *the derived group G_4^{der} is the maximal one allowed by the abelian type.*

Proof: We can assume that $(G_1,\mathcal{X}_1)$ is not unitary. This Proposition is only a $\mathbb{Z}_{(p)}$ version of the results of Satake on embeddings between hermitian symmetric domains of classical Lie type (see [Sa1] and [Sa2]). A $\mathbb{Q}$–version of this Proposition is presented in [Va10, Subsects. 4.2 to 4.8 and Rm. 4.8.2 (c)]. The passage from $\mathbb{Q}$-versions to $\mathbb{Z}_{(p)}$-versions is standard and thus for the first version of this paper, it will not be detailed here. □

3.10. Proof of Theorem 1.5 (ii) and (iii). We know that Theorem 1.5 (i) holds, cf. Subsections 3.1 to 3.8. We now check that Theorem 1.5 (ii) and (iii) holds as well. We can assume that $(G_1,\mathcal{X}_1)$ is not unitary, cf. Subsection 3.1. If $p>2$, Then Theorem 1.5 (ii) and (iii) follows from Sections 3.2, 3.3, 3.5, and 3.8. We will use the notations of Propositions 1.4 and 3.9 in order to show that Theorem 1.5 (ii) and (iii) holds even if $p=2$. For the sake of uniformity, we will continue to work with an arbitrary prime p. The central isogenies $G^{\mathrm{der}}\to G_1$ and $G_4^{\mathrm{der}}\to G_1$ can be identified, cf. properties 1.4 (iv) and 3.9 (iv).

Let $\mathcal{N}_2$ and $\mathcal{N}_3$ be the integral canonical models of $(G_2,\mathcal{X}_2,H_2)$ and $(G_3,\mathcal{X}_3,H_3)$ (respectively), cf. [Va6, Thm. 1.3]. They are quasi-projective, cf. loc. cit. As the integral canonical model $\mathcal{N}$ of $(G,\mathcal{X},H)$ exists (cf. Theorem 1.5 (i)) and is quasi-projective, the integral canonical model $\mathcal{N}_4$ of $(G_4,\mathcal{X}_4,H_4)$ exists as well (cf. [Va6, Prop. 2.4.3 (a)]). From Proposition 2.4 we get that $\mathcal{N}_4$ is the normalization of the Zariski closure of $\mathrm{Sh}(G_4,\mathcal{X}_4)/H_4$ in $\mathcal{N}_{3,E(G_4,\mathcal{X}_4)_{(p)}}$ (via the closed embedding morphism $\pi_2:\mathrm{Sh}(G_4,\mathcal{X}_4)/H_4\hookrightarrow\mathrm{Sh}(G_3,\mathcal{X}_3)_{E(G_4,\mathcal{X}_4)}/H_3$).

Let $\mathcal{N}_1$ be the normalization of the Zariski closure of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ in $\mathcal{N}_2$ (via the closed embedding morphism $q_1 : \mathrm{Sh}(G_1, \mathcal{X}_1)/H_1 \hookrightarrow \mathrm{Sh}(G_2, \mathcal{X}_2)_{E(G_1,\mathcal{X}_1)}/H_2$). It is a normal integral model of $(G_1, \mathcal{X}_1, H_1)$ which has the extension property, cf. [Va1, Prop. 3.4.1]. As $\mathcal{N}_2$ is quasi-projective, $\mathcal{N}_1$ is also quasi-projective. Due to the extension properties enjoyed by $\mathcal{N}_1$ to $\mathcal{N}_4$ and the fact that $\mathcal{N}_2$ to $\mathcal{N}_4$ are healthy regular schemes (being regular and formally smooth over $\mathbb{Z}_{(p)}$), we have a commutative diagram

$$\begin{tikzcd}
\mathcal{N}_4 \arrow[r, "\pi_{2,p}"] \arrow[d, "\pi_{1,p}"'] & \mathcal{N}_{3,E(G_4,\mathcal{X}_4)_{(p)}} \arrow[d, "q_{3,p}"] \\
\mathcal{N}_{1,E(G_4,\mathcal{X}_4)_{(p)}} \arrow[r, "q_{1,p}"] & \mathcal{N}_{2,E(G_4,\mathcal{X}_4)_{(p)}}
\end{tikzcd}$$

of normal $E(G_4, \mathcal{X}_4)_{(p)}$-schemes. The following properties hold:

(v) the morphism $q_{3,p}$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_{2,E(G_4,\mathcal{X}_4)_{(p)}}$;

(vi) the morphisms $\pi_{2,p}$, $\pi_{1,p}$, and $q_{1,p}$ are pro-finite;

(vii) the generic fibre of $\pi_{1,p}$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_{1,E(G_4,\mathcal{X}_4)}$. .

The property (v) is implied by [Va6, Thm. 5.1 (b)]. Due to the properties (vi) and (vii), the image of $\pi_{1,p}$ is an open closed subscheme of $\mathcal{N}_{1,E(G_4,\mathcal{X}_4)_{(p)}}$. Due to property (v) we easily get that $\pi_{1,p}$ is in fact a pro-étale cover of its image; thus this image is a regular, formally smooth scheme over $E(G_4, \mathcal{X}_4)_{(p)}$ and thus also over $E(G_1, \mathcal{X}_1)_{(p)}$. The connected components of $\mathcal{N}_{1,E(G_4,\mathcal{X}_4)_{(p)}}$ are permuted transitively by $G_1(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. From the last three sentences we get that $\mathcal{N}_{1,E(G_4,\mathcal{X}_4)_{(p)}}$ is a regular, formally smooth scheme over $E(G_4, \mathcal{X}_4)_{(p)}$. This implies that $\mathcal{N}_1$ is a regular, formally smooth scheme over $E(G_1, \mathcal{X}_1)_{(p)}$. From this and [Va1, Cor. 3.4.4] we get that $\mathcal{N}_1$ is the integral canonical model of $(G_1, \mathcal{X}_1, H_1)$. Thus Theorem 1.5 (ii) holds. Moreover we have a functorial morphism $\mathcal{N} \to \mathcal{N}_1$ of $E(G_1, \mathcal{X}_1)_{(p)}$-schemes.

Let $\mathbb{Z}_{(p)}^{\mathrm{un}}$ be the maximal $\mathbb{Z}_{(p)}$-subalgebra of $\overline{\mathbb{Q}}$ such that $\mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}})$ is a pro-étale cover of $\mathrm{Spec}(\mathbb{Z}_{(p)})$. Both $E(G_4, \mathcal{X}_4)_{(p)}$ and $E(G, \mathcal{X})_{(p)}$ are $\mathbb{Z}_{(p)}$-subalgebras of $\mathbb{Z}_{(p)}^{\mathrm{un}}$ and moreover the connected components of $\mathcal{N}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ and $\mathcal{N}_{4,\mathbb{Z}_{(p)}^{\mathrm{un}}}$ can be canonically identified, cf. [Va6, Prop. 2.4.3 (c)]. As the connected components of $\mathcal{N}_{4,\mathbb{Z}_{(p)}^{\mathrm{un}}}$ are pro-étale covers of certain connected components of $\mathcal{N}_{1,\mathbb{Z}_{(p)}^{\mathrm{un}}}$, we get that there exists a connected component of $\mathcal{N}$ which is a pro-étale cover of an open closed subscheme of $\mathcal{N}_1$. As the connected components of $\mathcal{N}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$ (cf. [Va1, Lem. 3.3.2]), we get that $\mathcal{N}$ itself is a pro-étale cover of an open closed subscheme of $\mathcal{N}_1$. Thus theorem 1.5 (iii) holds as well. This ends the proof of Theorem 1.5. □

We have the following Corollary to Theorem 1.5.

3.11. Corollary. *Let $(G_1, \mathcal{X}_1, H_1)$ and $\mathcal{N}_1$ be as in Theorem 1.5. Let $q_2 : (G_2, \mathcal{X}_2, H_2) \to (G_1, \mathcal{X}_1, H_1)$ be a map of Shimura triples of abelian type that induces an isomorphism $(G_2^{\mathrm{ad}}, \mathcal{X}_2^{\mathrm{ad}}, H_2^{\mathrm{ad}}) \xrightarrow{\sim} (G_1, \mathcal{X}_1, H_1)$. Then the normalization $\mathcal{N}_2$ of $\mathcal{N}_1$ in the ring of fractions*

of $\mathrm{Sh}(G_2, \mathcal{X}_2)/H_2$ *is a pro-étale cover of an open closed subscheme of* $\mathcal{N}_1$*. Moreover,* $\mathcal{N}_2$ *together with the natural action of* $G_2(\mathbb{A}_f^{(p)})$ *on it, is the integral canonical model of* $(G_2, \mathcal{X}_2, H_2)$ *and it is quasi-projective. If the integral canonical model* $\mathcal{N}$ *of Theorem 1.5 is projective, then* $\mathcal{N}_2$ *is projective too.*

Proof: We will use the notations of Theorem 1.5. We consider the fibre product (cf. [Va1, Subsect. 2.4 and Rm. 3.2.7 3)])

$$\begin{array}{ccc} (G_3, \mathcal{X}_3, H_3) & \xrightarrow{\pi_2} & (G, \mathcal{X}, H) \\ {\scriptstyle\pi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\ (G_2, \mathcal{X}_2, H_2) & \xrightarrow{q_2} & (G_1, \mathcal{X}_1, H_1). \end{array}$$

Due to the property 1.4 (iv), we have $G_3^{\mathrm{der}} = G^{\mathrm{der}}$. Thus by applying [Va6, Prop. 2.4.3 (a) and (b)] to $(G_3, \mathcal{X}_3, H_3)$ and $(G, \mathcal{X}, H)$, we get that the normalization $\mathcal{N}_3$ of $\mathcal{N}$ in the ring of fractions of $\mathrm{Sh}(G_3, \mathcal{X}_3)/H_3$ together with the natural action of $G_3(\mathbb{A}_f^{(p)})$ on it, is the integral canonical model of $(G_3, \mathcal{X}_3, H_3)$.

Let $W(\mathbb{F})$ be the ring of Witt vectors with coefficients in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. We consider an arbitrary $\mathbb{Z}_{(p)}$-embedding $E(G_2, \mathcal{X}_2)_{(p)} \hookrightarrow W(\mathbb{F})$.

We can identify each connected component $\mathcal{C}_3$ of $\mathcal{N}_{3,W(\mathbb{F})}$ with a connected component $\mathcal{C}$ of $\mathcal{N}_{W(\mathbb{F})}$, cf. [Va6, Prop. 2.4.3 (c)]. Let $\mathcal{C}_2$ and $\mathcal{C}_1$ be the connected components of $\mathcal{N}_{2,W(\mathbb{F})}$ and $\mathcal{N}_{1,W(\mathbb{F})}$ (respectively) dominated by $\mathcal{C}_3$. The composite morphism $\mathcal{C}_3 = \mathcal{C} \to \mathcal{C}_1$ of pro-finite covers, is a pro-étale cover (cf. Theorem 1.5 (iii)). Thus $\mathcal{C}_2$ is a pro-étale cover of $\mathcal{C}_1$. As the connected components of $\mathcal{N}_{2,W(\mathbb{F})}$ are permuted transitively by $G_2(\mathbb{A}_f^{(p)})$ (cf. [Va1, Lem. 3.3.2]), by using $G_2(\mathbb{A}_f^{(p)})$-translates of $\mathcal{C}_2$ we get that $\mathcal{N}_{2,W(\mathbb{F})}$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_{1,W(\mathbb{F})}$. Thus $\mathcal{N}_2$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_1$. As $\mathcal{N}_1$ has the extension property, each closed subscheme of it which is flat over $E(G_1, \mathcal{X}_1)_{(p)}$ has also the extension property. From the last two sentences we get that the $E(G_2, \mathcal{X}_2)_{(p)}$-scheme $\mathcal{N}_2$ has the extension property, cf. [Va1, Rm. 3.2.3.1 6)].

It is easy to see that there exists a compact, open subgroup $\tilde{H}$ of $G_2(\mathbb{A}_f^{(p)})$ such that the morphism $\mathcal{N}_2 \to \mathcal{N}_2/\tilde{H}$ is a pro-étale cover. As $\mathcal{N}$ is quasi-projective, we easily get that $\mathcal{N}_2/\tilde{H}$ is a smooth, quasi-projective $E(G_2, \mathcal{X}_2)_{(p)}$-scheme. Thus $\mathcal{N}_2$ is the integral canonical model of $(G_2, \mathcal{X}_2, H_2)$ and it is quasi-projective.

If $\mathcal{N}$ is projective, then $\mathcal{N}_1$ is projective (cf. Theorem 1.5 (iii)); this implies that $\mathcal{N}_2$ is projective. $\square$

4. The proof of Theorem 1.6

In Subsection 4.1 we prove Theorem 1.6 (a) to (c). In Subsection 4.2 we prove Theorem 1.6 (d). Let $(G_1, \mathcal{X}_1)$ be a Shimura pair of abelian type. We assume that the group $G_{1,\mathbb{Q}_p}$ is unramified. Let H_1 be a hyperspecial subgroup of $G_1(\mathbb{Q}_p)$.

Let $G_{1,\mathbb{Z}_p}$ be the reductive group scheme over $\mathbb{Z}_p$ such that its generic fibre is $G_{1,\mathbb{Q}_p}$ and we have $H_1 = G_{1,\mathbb{Z}_p}(\mathbb{Z}_p)$. Let $H_1^{\mathrm{ad}} := G_{1,\mathbb{Z}_p}^{\mathrm{ad}}(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $G_1^{\mathrm{ad}}(\mathbb{Q}_p)$.

If the adjoint group G_1^{ad} is trivial, then Theorem 1.6 is well known (it is an easy consequence of either [Mi2, Rm. 2.16] or [Va1, Example 3.2.8]). Thus to prove Theorem 1.6 we can assume that the adjoint group G_1^{ad} is non-trivial.

4.1. Proofs of 1.6 (a) to (c). Let $(G_1^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}, H_1^{\mathrm{ad}}) = \prod_{i\in I}(G_i, \mathcal{X}_i, H_i)$ be the product decomposition into simple, adjoint Shimura triples. Thus each $(G_i, \mathcal{X}_i)$ is a simple, adjoint Shimura pair. Let $\mathcal{N}_i$ be the integral canonical model of $(G_i, \mathcal{X}_i, H_i)$, cf. Theorem 1.5 (ii). We consider the product $\mathcal{N}^{\mathrm{ad}} := \prod_{i\in I} \mathcal{N}_{i,E(G_1^{\mathrm{ad}},\mathcal{X}_1^{\mathrm{ad}})_{(p)}}$ of $E(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})_{(p)}$-schemes; it is the integral canonical model of $(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}}, H_1^{\mathrm{ad}})$. Let $\mathcal{N}_1$ be the normalization of $\mathcal{N}^{\mathrm{ad}}$ in the ring of fractions of $\mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$. We check that:

(*) the natural morphism $m_1 : \mathcal{N}_1 \to \mathcal{N}^{\mathrm{ad}}$ of $E(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})_{(p)}$-schemes is pro-finite and a pro-étale cover of its image.

Let $q_3 : (G_3, \mathcal{X}_3, H_3) \to (G_1, \mathcal{X}_1, H_1)$ be a cover such that at the level of reflex fields we have $E(G_3, \mathcal{X}_3) = E(G, \mathcal{X})$ and the semisimple group cover G_3^{der} of G_1^{der} is the maximal one allowed by the abelian type, cf. [Va1, Rm. 3.2.7 10)]. Similarly we consider a cover $q_{3,i}$: $(G_{3,i}, \mathcal{X}_{3,i}, H_{3,i}) \to (G_i, \mathcal{X}_i, H_i)$ such that at the level of reflex fields we have $E(G_{3,i}, \mathcal{X}_{3,i}) = E(G_i, \mathcal{X}_i)$ and the semisimple group cover $G_{3,i}^{\mathrm{der}}$ of G_i is the maximal one allowed by the abelian type. The morphisms $\mathrm{Sh}(G_3, \mathcal{X}_3)/H_3 \to \mathrm{Sh}(G_1, \mathcal{X}_1)/H_1$ and $\mathrm{Sh}(G_{3,i}, \mathcal{X}_{3,i})/H_{3,i} \to \mathrm{Sh}(G_i, \mathcal{X}_i)/H_i$ are pro-étale covers, cf. [Va6, Lem. 2.3.1]. In particular, we get that to check that the property (*) holds, we can assume that G_1^{der} is the maximal semisimple group cover of G_1^{ad} allowed by the abelian type. Let $(G_4, \mathcal{X}_4, H_4) := \prod_{i\in \mathcal{J}}(G_{3,i}, \mathcal{X}_{3,i}, H_{3,i})$. We have $(G_4^{\mathrm{ad}}, \mathcal{X}_4^{\mathrm{ad}}, H_4^{\mathrm{ad}}) = (G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}}, H_1^{\mathrm{ad}})$ and $G_4^{\mathrm{der}} = G_1^{\mathrm{der}}$. Based on [Va6, Prop. 2.3.3 (a) and (c)], to prove that the property (*) holds we can also assume that we have $(G_4, \mathcal{X}_4, H_4) = (G_1, \mathcal{X}_1, H_1)$. Thus to check that the property (*) holds, we can assume that the set I has one element (i.e., G_1^{ad} is a simple, adjoint group over $\mathbb{Q}$). But this case follows from Corollary 3.11.

As in the end of the proof of Corollary 3.11 we argue that $\mathcal{N}_1$ is the integral canonical model of $(G_1, \mathcal{X}_1, H_1)$ and it is quasi-projective. Thus Theorem 1.6 (a) holds. Theorem 1.6 (b) follows from the property (*) applied to the morphisms $m_1 : \mathcal{N}_1 \to \mathcal{N}^{\mathrm{ad}}$ and $m_2 : \mathcal{N}_2 \to \mathcal{N}^{\mathrm{ad}}$ of $E(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})_{(p)}$-schemes, once we remark that m_1 is the composite of the functorial morphism $\mathcal{N}_2 \to \mathcal{N}_1$ of $E(G_2, \mathcal{X}_2)_{(p)}$-schemes with m_2.

Theorem 1.6 (c) follows from Theorem 1.6 (a) and Proposition 2.4. $\square$

4.2. Proof of 1.6 (d). Let the following notations $(G_1^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}}, H_1^{\mathrm{ad}}) = \prod_{i\in I}(G_i, \mathcal{X}_i, H_i)$, $\mathcal{N}_i$, $\mathcal{N}^{\mathrm{ad}} := \prod_{i\in I} \mathcal{N}_{i,E(G_1^{\mathrm{ad}},\mathcal{X}_1^{\mathrm{ad}})_{(p)}}$, and $\mathcal{N}_1$ be as Subsection 4.1. To prove that $\mathcal{N}_1$ is projective, it suffices to show that each $\mathcal{N}_i$ is projective. Thus we can assume that G_1 is a simple, adjoint group over $\mathbb{Q}$. Therefore we can appeal to the (notations of) Theorem 1.5. The connected components of $\mathcal{N}_1$ are permuted transitively by $G_1(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. Based on this and Theorem 1.5 (iii), to prove that $\mathcal{N}_1$ is projective it suffices to show that $\mathcal{N}$ is projective. Let $N \geq 3$ be a positive integer relatively prime to p. Let $K(N)_p$ be the open, closed subgroup of $\mathbf{GSp}(L, \psi)(\mathbb{A}_f^p)$ such that $K(N) := K_p \times K(N)_p$ is the subgroup of $\mathbf{GSp}(L, \psi)(\widehat{\mathbb{Z}})$ formed by elements congruent to the identity modulo N. Let $H(N)_p := G(\mathbb{A}_f^{(p)}) \cap K(N)_p$. Then $H(N) := H_p \times H(N)_p = G(\mathbb{A}_f) \cap K(N)$.

As $N \geq 3$, a principally polarized abelian scheme with level-N structure has no automorphism (see [Mu, Ch. IV, 21, Thm. 5] for this result of Serre). This implies that $K(N)$ acts freely on $\mathcal{A}_{d,1,N}$. From this we get that $H(N)$ acts freely on $\mathcal{N}$. Therefore $\mathcal{N}$ is a pro-étale cover of the $E(G,\mathcal{X})_{(p)}$-scheme $\mathcal{N}_N := \mathcal{N}/H(N)$. The scheme $\mathcal{N}_N$ is the normalization of $\mathcal{A}_{d,1,N,\mathbb{Z}_{(p)}}$ in $\mathrm{Sh}(G,\mathcal{X})/H(N)$ and therefore it is a finite $\mathcal{A}_{d,1,N,\mathbb{Z}_{(p)}}$-scheme. Therefore $\mathcal{N}_N$ is a quasi-projective $E(G,\mathcal{X})_{(p)}$-scheme. Thus the $E(G,\mathcal{X})_{(p)}$-scheme $\mathcal{N}_N$ is projective if and only if $\mathcal{N}$ is projective.

If the condition 1.6 (d.i) (resp. 1.6 (d.ii)) holds, then the projectiveness of $\mathcal{N}_N$ is implied by either [Pa] or [Va5, Lem. 2.3.1] (resp. by [Va5, Cor. 4.3]). If the condition 1.6 (d.iii) holds, then the fact that $\mathcal{N}$ is projective is implied by [Va6, Thm. 5.1 (c)]. Therefore, regardless of which one of the three conditions 1.6 (d.i) to (d.iii) holds, we get that $\mathcal{N}$ is projective. Thus Theorem 1.6 (d) holds. This ends the proof of Theorem 1.6. $\square$

Appendix: A complement on a motivic conjecture of Milne for 2-divisible groups

Let $p = 2$. Let k be an algebraically closed field of characteristic 2. Let $W(k)$ be the ring of Witt vectors with coefficients in k. In this Appendix we prove a variant of a motivic conjecture of Milne in the context of 2-divisible groups over $W(k)$. More precisely, we solve the Problem of [Va7, Appendix, Subsect. B5]. Let $B(k) := W(k)[\frac{1}{p}]$ be the field of fractions of $W(k)$. Let σ be the Frobenius automorphism of k, $W(k)$, and $B(k)$. We fix an algebraic closure $\overline{B(k)}$ of $B(k)$. Let $\mathrm{Gal}(B(k)) := \mathrm{Gal}(\overline{B(k)}/B(k))$.

Let D_k be a p-divisible group over k. Let (M,ϕ) be the contravariant Dieudonné module of D_k. Thus M is a free $W(k)$-module of rank equal to the height of D and $\phi : M \to M$ is a σ-linear endomorphism such that we have $pM \subseteq \phi(M)$. Let F^1 be a direct summand of M such that we have $\phi(M + \frac{1}{p}F^1) = M$. The rank of F^1 is the dimension of D_k. Let $M^* := \mathrm{Hom}(M, W(k))$. Let $\mathcal{T}(M) := \oplus_{s,t\in\mathbb{N}\cup\{0\}} M^{\otimes s} \otimes_{W(k)} M^{*\otimes t}$. Let $F^0 := M$ and $F^2 := 0$. Let $F^{1,*} := 0$, $F^{0,*} := \{y \in M^* | y(F^1) = 0\}$, and $F^{-1,*} := M^*$. Let $(F^i(\mathcal{T}(M)))_{i\in\mathbb{Z}}$ be the tensor product filtration of $\mathcal{T}(M)$ defined by the exhaustive, separated filtrations $(F^i)_{i\in\{0,1,2\}}$ and $(F^{i,*})_{i\in\{-1,0,1\}}$ of M and M^* (respectively). We refer to $(F^i(\mathcal{T}(M)))_{i\in\mathbb{Z}}$ as the filtration of $\mathcal{T}(M)$ defined by F^1. For $f \in M^*[\frac{1}{p}]$ let $\phi(f) := \sigma \circ f \circ \phi^{-1} \in M^*[\frac{1}{p}]$. Thus ϕ acts in the usual tensor product way on $\mathcal{T}(M[\frac{1}{p}])$. In particular, ϕ acts on $\mathrm{End}(M[\frac{1}{p}]) = M[\frac{1}{p}] \otimes_{B(k)} M^*[\frac{1}{p}]$ via the rule: an element $e \in \mathrm{End}(M[\frac{1}{p}])$ is mapped to $\phi(e) := \phi \circ e \circ \phi^{-1}$.

We assume that there exists a family $(t_\alpha)_{\alpha\in\mathcal{J}}$ of tensors of $F^0(\mathcal{T}(M))[\frac{1}{p}]$ such that the following two properties hold:

(i) we have $\phi(t_\alpha) = t_\alpha$ for all $\alpha \in \mathcal{J}$;

(ii) the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$ is a reductive group scheme $\mathcal{G}$ over $W(k)$.

Following [Va12] and [Va13], we refer to the quadruple $(M, F^1, \phi, \mathcal{G})$ as a *filtered Shimura F-crystal* over k. Due to properties (i) and (ii), we have

(iii) $\phi(\mathrm{Lie}(\mathcal{G}_{B(k)})) = \mathrm{Lie}(\mathcal{G}_{B(k)})$ i.e., the pair $(\mathrm{Lie}(\mathcal{G}_{B(k)}), \phi)$ is a Lie F-isocrystal of $(\mathrm{End}(M)[\frac{1}{p}], \phi)$.

Let D be a 2-divisible group over $W(k)$ whose filtered Dieudonné module is the triple (M, F^1, ϕ); its special fibre is D_k. Let D^{t} be the Cartier dual of D.

Let $H^1(D) := T_2(D^{\mathrm{t}}_{B(k)})(-1)$ be the dual of the Tate-module $T_2(D_{B(k)})$ of $D_{B(k)}$. Thus $H^1(D)$ is a free $\mathbb{Z}_2$-module of the same rank as M and $\mathrm{Gal}(B(k))$ acts on it. Let $\mathcal{T}(H^1(D)) := \oplus_{s,t\in\mathbb{N}\cup\{0\}}[H^1(D)]^{\otimes s} \otimes_{\mathbb{Z}_2} [H^1(D)^*]^{\otimes t}$. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be the family of étale Tate-cycles on $D_{B(k)}$ that corresponds to $(t_\alpha)_{\alpha\in\mathcal{J}}$ via Fontaine comparison theory for D. Each v_α is a tensor of $\mathcal{T}(H^1(D)[\frac{1}{p}])$ fixed by $\mathrm{Gal}(B(k))$. Let $\mathcal{G}_{\acute{e}t}$ be the Zariski closure in $\mathbf{GL}_{H^1(D)}$ of the reductive subgroup of $\mathbf{GL}_{H^1(D)[\frac{1}{p}]}$ that fixes v_α for all $\alpha \in \mathcal{J}$.

We denote by $R^\wedge$ the 2-adic completion of a $W(k)$-algebra R. The goal of this Appendix is to prove the following Theorem which for $p = 2$ complements the motivic Conjecture of Milne proved for $p > 2$ in [Va7] (this Theorem was stated as a Problem in [Va7, Appendix, Subsect. B5]).

A1. Theorem. *Giving the quadruple $(M, F^1, \phi, (t_\alpha)_{\alpha\in\mathcal{J}}$, we can choose the 2-divisible group D over $W(k)$ such that there exists an isomorphism*

$$r_D : (M, (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (H^1(D) \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha\in\mathcal{J}}).$$

Proof: Let $\mu : \mathbb{G}_m \to \mathbf{GL}_M$ be the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512]. It fixes each t_α, cf. the functorial aspects of [Wi, p. 513]. Thus μ factors as a cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$.

Let n be the smallest positive integer such that all Newton polygon slopes of (M, ϕ) belong to $\frac{1}{n}\mathbb{Z} \cap \mathbb{Q}$. We consider the direct sum decomposition

$$(M[\frac{1}{p}], \phi) = \oplus_{\gamma\in[0,1]\cap\mathbb{Q}}(W_\gamma, \phi)$$

into F-isocrystals over k with the property that for each $\gamma \in [0, 1] \cap \mathbb{Q}$, all Newton polygon slopes of (W_γ, ϕ) are γ. Let $\nu : \mathbb{G}_m \to \mathbf{GL}_{M[\frac{1}{p}]}$ be the Newton polygon cocharacter of $(M[\frac{1}{p}], \phi)$: it is defined by the property that $\mathbb{G}_m$ acts through ν on W_γ via the $-n\gamma^{\mathrm{th}}$ power of the identity character of $\mathbb{G}_m$. Due to the property (i) above, ν fixes each t_α. Thus ν factors as a cocharacter $\nu : \mathbb{G}_m \to \mathcal{G}_{B(k)}$.

We consider the following two cases.

Case 1: the basic case. We assume that the triple $(M, \phi, \mathcal{G})$ is *basic* i.e., all Newton polygons slopes of $(\mathrm{Lie}(\mathcal{G}_{B(k)}), \phi)$ are 0. This means that $\nu : \mathbb{G}_m \to \mathcal{G}_{B(k)}$ factors through the identity component $Z^0(\mathcal{G}_{B(k)})$ of the center $Z(\mathcal{G}_{B(k)})$ of $\mathcal{G}_{B(k)}$. Therefore ν extends to a cocharacter $\nu : \mathbb{G}_m \to Z^0(\mathcal{G})$, where $Z^0(\mathcal{G})$ is the maximal torus of the center $Z(\mathcal{G})$ of $\mathcal{G}$. This implies that the following three properties hold:

(iv) we have a direct sum decomposition $M = \oplus_{\gamma\in[0,1]\cap\mathbb{Q}}M_\gamma$, where $M_\gamma := M \cap W_\gamma$;

(v) the cocharacters μ and ν of $\mathcal{G}$ commute and therefore we have a direct sum decomposition $F^1 = \oplus_{\gamma\in[0,1]\cap\mathbb{Q}}F^1_\gamma$, where $F^1_\gamma := F^1 \cap M_\gamma$;

(vi) the group scheme $\mathcal{G}$ is a closed subgroup scheme of $\prod_{\gamma\in[0,1]\cap\mathbb{Q}}\mathbf{GL}_{M_\gamma}$.

We have a direct sum decomposition $(M,F^1,\phi)=\oplus_{\gamma\in[0,1]\cap\mathbb{Q}}(M_\gamma,F^1_\gamma,\phi)$, cf. property (v). Let D_γ be the unique 2-divisible group over $W(k)$ whose filtered Dieudonné module is $(M_\gamma,F^1_\gamma,\phi)$, cf. [Fo, Ch. IV, Prop. 1.6] (strictly speaking, [Fo, Ch. IV, Prop. 1.6] is stated in terms of *Honda triples* $(M,\phi(\frac{1}{p}F^1),\phi)$ and not in terms of filtered Dieudonné modules (M,F^1,ϕ)). We take D such that we have a product decomposition

$$D=\prod_{\gamma\in[0,1]\cap\mathbb{Q}}D_\gamma. \tag{9}$$

Let $\mathcal{H}$ be the smallest reductive, closed subgroup scheme of $\mathcal{G}$ such that the following two properties hold:

(vii) we have $\phi(\mathrm{Lie}(\mathcal{H}_{B(k)}))=\mathrm{Lie}(\mathcal{H}_{B(k)})$, the cocharacter $\mu:\mathbb{G}_m\to\mathcal{G}$ factors through $\mathcal{H}$, and $Z^0(\mathcal{G})$ is a torus of the center $Z(\mathcal{H})$ of $\mathcal{H}$.

Due to the property (vi) and the smallest property of $\mathcal{G}$, $\mathcal{H}$ is a closed subgroup scheme of $Z^1(\mathbf{GL}_{M_0})\times_{W(k)}[\prod_{\gamma\in(0,1)\cap\mathbb{Q}}\mathbf{GL}_{M_\gamma}]\times_{W(k)}Z(\mathbf{GL}_{M_1})$, where $Z^1(\mathbf{GL}_{M_0})$ is the trivial subgroup scheme of $\mathbf{GL}_{M_0}$ and $Z(\mathbf{GL}_{M_1})$ is the center of $\mathbf{GL}_{M_1}$.

Thus by replacing $\mathcal{G}$ with $\mathcal{H}$, we can assume that $\mathcal{G}$ acts trivially on M_0 and via scalar automorphisms on M_1.

Let $\vartheta:=p1_M\circ\phi^{-1}:M\to M$ be the Verschiebung map of (M,ϕ). For $g\in\mathcal{G}(W(k))$, by the *D-truncation mod p* of $(M,g\phi,\mathcal{G})$ we mean the quadruple $(\bar{M},\bar{g}\bar{\phi},\bar{\vartheta}\bar{g}^{-1},\mathcal{G}_k)$, where $\bar{\phi},\bar{\theta}:\bar{M}\to\bar{M}$ are the reductions modulo 2 of $\phi,\vartheta:M\to M$ and where $\bar{g}\in\mathcal{G}(k)$ is the reduction modulo 2 of g. Let $\mathcal{O}$ be the reduced, locally closed subscheme of $\mathcal{G}_k$ such that we have $\bar{g}\in\mathcal{O}$ if and only if there exists an element $\bar{h}\in\mathcal{G}(k)$ that defines an isomorphism between $(\bar{M},\bar{\phi},\bar{\vartheta},\mathcal{G}_k)$ and $(\bar{M},\bar{g}\bar{\phi},\bar{\vartheta}\bar{g}^{-1},\mathcal{G}_k)$. It is an orbit of a suitable action of a smooth, connected, affine group on $\mathcal{G}_k$ (see [Va11, Subsect. 5.1]) and therefore it is a connected, regular scheme over k. We choose an element $g\in\mathcal{G}(W(k))$ such that the following three properties hold (cf. [Va11, Cor. 11.1 (c) and (d)]):

(viii) its reduction modulo 2 is not $1_{\bar{M}}$;

(ix) there exists a smooth, connected, affine curve Y_k inside $\mathcal{O}$ which passes through $1_{\bar{M}}$ and $\bar{g}$ and which is equipped with an étale morphism $m_k:Y_k\to\mathrm{Spec}(k[t])$;

(x) there exists a maximal torus $\mathcal{T}$ of $\mathcal{G}$ through which μ factors and whose Lie algebra is normalized by $g\phi$.

Let $m:Y=\mathrm{Spec}(R)\to\mathrm{Spec}(W(k)[t])$ be an étale morphism between smooth, affine curves over $W(k)$ that lifts $m_k:Y_k\to\mathrm{Spec}(k[t])$; via it, we will view t as an element of R. As in [Va7, Subsect. 3.1] we argue that we can assume that $R^\wedge$ is equipped with a Frobenius lift that takes $t\in R$ to t^2 and for which the two sections $\mathrm{Spec}(W(k))\to\mathrm{Spec}(R^\wedge)$ that define 1_M and g are Teichmüller lifts. As in [Va7, Sect. 3] we argue that to prove the Theorem we can replace the quintuple $(M,F^1,\phi,\mathcal{G},D=\prod_{\gamma\in[0,1]\cap\mathbb{Q}}D_\gamma)$ by the quintuple $(M,F^1,g\phi,\mathcal{T},\tilde{D}=\prod_{\gamma\in[0,1]\cap\mathbb{Q}}D_{g,\gamma})$. This makes sense even if $p=2$ as in all steps of loc. cit. we can work with 2-divisible groups which are direct sums of étale 2-divisible

groups, of 2-divisible groups of multiplicative type, and of 2-divisible groups whose fibres in characteristic 2 are all connected.

Therefore to prove the Theorem we can assume that $\mathcal{T} = \mathcal{G}$. Due to the decomposition (9), as in [Va7, Lem. 4.1.1] one argues that $\mathcal{G}_{\acute{e}t}$ is a torus. Thus in this Case 1, the Theorem follows from [Va7, Thm. 1.2].

Case 2: reduction to the non-basic case. We assume that the triple $(M, \phi, \mathcal{G})$ is not basic i.e., not all Newton polygons slopes of $(\mathrm{Lie}(\mathcal{G}_{B(k)}), \phi)$ are 0. Let $P_{\mathcal{G}}^{+}(\phi)$ be the parabolic subgroup scheme of $\mathcal{G}$ such that $\mathrm{Lie}(P_{\mathcal{G}}^{+}(\phi)_{B(k)})$ is the maximal direct summand of $\mathrm{Lie}(\mathcal{G}_{B(k)})$ which is left invariant by ϕ and such that all Newton polygon slopes of $(\mathrm{Lie}(P_{\mathcal{G}}^{+}(\phi)_{B(k)}), \phi)$ are 0, cf. [Va12, Lem. 2.3.1]. Let $U_{\mathcal{G}}^{+}(\phi)$ be the unipotent radical of $P_{\mathcal{G}}^{+}(\phi)$.

Using a local deformation of $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \mathcal{G})$ as in [Va8, Appendix B, Thm. B7.4] and a natural analogue of [Va8, Lem. 3.5.2] for it, to prove the Theorem we can replace F^1 by any other direct summand F_1^1 of M for which we have $t_\alpha \in F_1^0(\mathcal{T}(M))[\frac{1}{p}]$ for all elements $\alpha \in \mathcal{J}$, where $(F_1^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ is the filtration of $\mathcal{T}(M)$ defined by F_1^1. Thus the quintuple $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \mathcal{G})$ has the same properties as $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \mathcal{G})$. Based on [Va12, Thm. 3.1.2] we can assume that there exist:

(xi) a Levi subgroup $\mathcal{L}$ of $P_{\mathcal{G}}^{+}(\phi)$ such that μ factors through a maximal torus $\mathcal{T}$ of $\mathcal{L}$;

(xii) an element $u \in U_{\mathcal{G}}^{+}(\phi)(W(k))$ such that the quadruple $(M, F^1, u\phi, \mathcal{L})$ is a filtered Shimura F-crystal over k.

Based on the property (xii), there exists a family of tensors $(t_\alpha)_{\alpha \in \tilde{\mathcal{J}}}$ of $\mathcal{T}(M[\frac{1}{p}])$ fixed by ϕ which is indexed by a set $\tilde{\mathcal{J}}$ that contains $\mathcal{J}$, which enlarges the family $(t_\alpha)_{\alpha \in \mathcal{J}}$, and for which $\mathcal{L}_{B(k)}$ is the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \tilde{\mathcal{J}}$.

We can assume that the element u is not congruent to 1_M modulo 2. Let $\tilde{D}$ be the unique 2-divisible group over $W(k)$ whose filtered Dieudonné module is $(M, F^1, u\phi)$ and which is a direct sum of 2-divisible groups over $W(k)$ whose special fibres have only one Newton polygon slope (i.e., and which is a direct sum as in (9)). Based on the Case 1, Theorem holds for the quintuple $(M, F^1, u\phi, (t_\alpha)_{\alpha \in \tilde{\mathcal{J}}}, \tilde{D})$.

Let Y be $U_{\mathcal{G}}^{+}(\phi)$ but viewed only as a $W(k)$-scheme. We write $Y = \mathrm{Spec}(R)$, where $R = W(k)[x_1, \ldots, u_l]$ is a polynomial $W(k)$-algebra in l variables (l being the relative dimension of $U_{\mathcal{G}}^{+}(\phi)$). Let $u_R \in U_{\mathcal{G}}^{+}(\phi)(R)$ be the universal element.

As in [Va7, Subsect. 3.1] we argue that we can assume that there exists a Frobenius lift Φ_R of R that takes each $x_i \in R$ to x_i^2 and for which the two sections $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^\wedge)$ that define the elements 1_M and u of $U_{\mathcal{G}}^{+}(\phi)(W(k))$ are Teichmüller lifts. As in [Va7, Thm. 3.2] one argues that there is an affine, pro-étale morphism $\ell : \mathrm{Spec}(Q_\infty) \to \mathrm{Spec}(R)$ such that the following three properties hold:

(xiii) the k-scheme $Q_\infty/2Q_\infty$ is connected;

(xiv) the morphism $\mathrm{Spec}(W(k)) \to Y$ that defines the element 1_M (resp. u) of $U_{\mathcal{G}}^{+}(\phi)(W(k))$ factors as a morphism $a_1 : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_\infty)$ (resp. factors uniquely as a morphism $a_0 : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_\infty)$);

(xv) the extension of $(M \otimes_{W(k)} R^\wedge, u_R(\phi \otimes \Phi_R), (t_\alpha)_{\alpha \in \tilde{\mathfrak{J}}})$ to Q_∞ is equipped with an integrable, nilpotent modulo 2 connection ∇_∞ that annihilates each tensor t_α with $\alpha \in \tilde{\mathfrak{J}}$.

Let Φ_Q be the Frobenius lift of $Q := Q_\infty^\wedge$ defined naturally by Φ_R. The k-algebra $Q/2Q$ has a finite 2-basis. Therefore, based on this and [BM, Prop. 1.3.3], we will identify a (filtered) F-crystal over k with its evaluation (viewed with connection) at the thickening associated naturally to the closed embedding $\mathrm{Spec}(Q/2Q) \hookrightarrow \mathrm{Spec}(Q)$. Here and in all that follows, such evaluations will be viewed with connections. From the property (xiv) we get that the quadruple

$$(M \otimes_{W(k)} Q, F^1 \otimes_{W(k)} Q, (u_R \circ \ell)(\phi \otimes \Phi_Q), \nabla_\infty, (t_\alpha)_{\alpha \in \tilde{\mathfrak{J}}})$$

is a filtered F-crystal over $\mathrm{Spec}(Q/2Q)$ equipped with a family of crystalline tensors. For each integer $i \in \{0, 1\}$, we denote also by $a_i : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q)$ the morphism defined naturally by a_i. We have (cf. [Va7, Thm. 3.4.1 (a)]):

A1.1. Proposition. *There exists a unique 2-divisible group $\mathcal{D}_{Q/2Q}$ over Spec$(Q/2Q)$ for which we have $a_{0,k}^*(\mathcal{D}_{Q/2Q}) = \tilde{D}_k$ and whose filtered F-crystal over Spec$(Q/2Q)$ is defined by the triple $(M \otimes_{W(k)} Q, (u_R \circ \ell)(\phi \otimes \Phi_Q), \nabla_\infty)$.*

Let $\tilde{\mathcal{D}}_{Q/4Q}$ be a 2-divisible group over $Q/4Q$ that lifts $\mathcal{D}_{Q/2Q}$, cf. [Il, Thm. 4.4 a) and f)]. Let $\delta(2)^{\mathrm{tr}}$ be the trivial divided power structure of the ideal (2) of $Q/4Q$ defined by the identities $(2)^{[s]} = 0$ with $s \in \mathbb{N}$, $s \geq 2$. Let $\tilde{F}^{-1}(\mathrm{End}(M))$ be the maximal direct summand of $\mathrm{End}_{W(k)}(M)$ on which $\mathbb{G}_m$ acts via μ as the identity character of $\mathbb{G}_m$. Let $\mathcal{L}_{\text{crys-lift}}$ (resp. $\mathcal{L}_{\text{lift}}$) be the free $Q/2Q$-module of lifts of $F^1 \otimes_{W(k)} Q/2Q$ to direct summands of $M \otimes_{W(k)} Q/4Q$, the zero element corresponding to the Hodge filtration defined by $\tilde{\mathcal{D}}_{Q/4Q}$ and by the standard divided power structure (resp. and by $\delta(2)^{\mathrm{tr}}$) of the ideal (2) of $Q/4Q$. The $Q/2Q$-module structure of $\mathcal{L}_{\text{crys-lift}}$ (resp. of $\mathcal{L}_{\text{lift}}$) is defined naturally by identifying $\mathcal{L}_{\text{crys-lift}}$ (resp. $\mathcal{L}_{\text{lift}}$) with the set of images of the lift of $M \otimes_{W(k)} Q/4Q$ that defines the zero element of $\mathcal{L}_{\text{crys-lift}}$ (resp. of $\mathcal{L}_{\text{lift}}$) through elements of the form $1_{M \otimes_{W(k)} Q/4Q} + 2u \in \mathbf{GL}_M(Q/4Q)$, where $u \in \tilde{F}^{-1}(\mathrm{End}_{W(k)}(M)) \otimes_{W(k)} Q/4Q$. Let $y \in \mathcal{L}_{\text{crys-lift}}$ be such that it corresponds to $F^1 \otimes_{W(k)} Q/4Q$.

We define a natural map of sets

$$\mathcal{E}(Q/2Q) : \mathcal{L}_{\text{lift}} \to \mathcal{L}_{\text{crys-lift}}$$

as follows. Let $x \in \mathcal{L}_{\text{lift}}$ and let $\mathcal{D}^x_{Q/4Q}$ be the lift of $\mathcal{D}_{Q/2Q}$ defined by x and Grothendieck–Messing deformation theory (the divided power structure of the ideal (2) of $Q/4Q$ being $\delta(2)^{\mathrm{tr}}$). We define $\mathcal{E}(Q/2Q)(x)$ to be the Hodge filtration of $M \otimes_{W(k)} Q/4Q$ defined by $\mathcal{D}^x_{Q/4Q}$ using the standard divided power structure of the ideal (2) of $Q/4Q$. This map of sets is defined naturally by a morphism

$$\mathcal{E} : \mathbb{A}^{d^2}_{Q/2Q} \to \mathbb{A}^{d^2}_{Q/2Q}$$

of $Q/2Q$-schemes. Due to the property (xiv), the 2-rank a of pull backs of $\mathcal{D}_{Q/2Q}$ via geometric points is the same as the 2-rank of D_k. It is well known that this implies that

all the geometric fibres of $\mathcal{E}$ are finite and have the same number of elements (equal to 2^{a^2}). Therefore $\mathcal{E}$ is a finite morphism.

It is also well known that the geometric fibres of $\mathcal{E}$ are étale. We conclude that $\mathcal{E}$ is a finite, étale morphism. The map $\mathcal{E}(Q/2Q)$ has a functorial aspect with respect to pulls back of $\mathcal{D}_{Q/2Q}$ via either geometric points of $\mathrm{Spec}(Q/2Q)$ or affine, étale schemes over $\mathrm{Spec}(Q/2Q)$. Thus, up to a replacement of $\mathrm{Spec}(Q)$ by a connected, étale cover of it, we can assume that the set $\mathcal{E}(Q/2Q)^{-1}(y)$ has 2^{a^2} points. Based on this we conclude that:

A1.2. Proposition. *There exists a unique* 2*-divisible group* $\mathcal{D}$ *over* Spec(Q) *for which we have* $a_0^*(\mathcal{D}) = \tilde{D}$ *and whose filtered* F*-crystal over* Spec$(Q/2Q)$ *is defined by the quadruple* $(M \otimes_{W(k)} Q, F^1 \otimes_{W(k)} Q, (u_R \circ \ell)(\phi \otimes \Phi_Q), \nabla_\infty)$.

Let $D := a_1^*(\mathcal{D})$. Due to Proposition A1.2, as in [Va7, Subsect. 3.5] we argue that to prove the Theorem in the context of the quintuple $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, D)$ it suffices to prove it in the context of the quintuple $(M, F^1, u\phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \tilde{D})$. But we know that the Theorem holds in the context of the quintuple $(M, F^1, u\phi, (t_\alpha)_{\alpha \in \tilde{\mathcal{J}}}, \tilde{D})$. As we have $\mathcal{J} \subseteq \tilde{\mathcal{J}}$, from the last two sentences we get that the Theorem holds. $\square$

A2. Remark. Suppose that (M, ϕ) has a principal quasi-polarization ψ_M such that we have $\psi_M(F^1 \otimes F^1) = 0$ and the group scheme $\mathcal{G}$ normalizes the $W(k)$-span of ψ_M. Then in Theorem A1 we can choose D such that moreover the following condition holds:

(i) ψ_M is the crystalline realization of a principal quasi-polarization λ_D of D;

To check this, we first remark that if D is as in the decomposition (9), then the condition (i) holds (cf. [Fo, Ch. IV, Prop. 1.6]). Based on this we easily get that in Theorem A1 we can choose D such that moreover the condition (i) holds. From this and [Va7, Rm. 4.4 (a)] we get that:

(ii) there exists an isomorphism $r_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H^1(D) \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$ that takes ψ_M to the perfect alternating form on $H^1(D)$ defined by λ_D.

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